

INTRODUCTION TO VERTEX ALGEBRAS, BORCHERDS ALGEBRAS, AND THE MONSTER LIE ALGEBRA^{*}

Reinhold W. Gebert[†]

IIInd Institute for Theoretical Physics, University of Hamburg
Luruper Chaussee 149, D-22761 Hamburg, Germany

August 25, 1993

The theory of vertex algebras constitutes a mathematically rigorous axiomatic formulation of the algebraic origins of conformal field theory. In this context Borcherds algebras arise as certain “physical” subspaces of vertex algebras. The aim of this review is to give a pedagogical introduction into this rapidly-developing area of mathematics. Based on the machinery of formal calculus we present the axiomatic definition of vertex algebras. We discuss the connection with conformal field theory by deriving important implications of these axioms. In particular, many explicit calculations are presented to stress the eminent role of the Jacobi identity axiom for vertex algebras. As a class of concrete examples the vertex algebras associated with even lattices are constructed and it is shown in detail how affine Lie algebras and the fake Monster Lie algebra naturally appear. This leads us to the abstract definition of Borcherds algebras as generalized Kac-Moody algebras and their basic properties. Finally, the results about the simplest generic Borcherds algebras are analysed from the point of view of symmetry in quantum theory and the construction of the Monster Lie algebra is sketched.

^{*}to appear in Int. J. Mod. Phys.

[†]Supported by Konrad-Adenauer-Stiftung e.V.

1 Introduction

Nowadays most theoretical physicists are aware of the fact that the present relation between mathematics and physics is characterized by an increasing overlap between them. Prominent examples for that development are the vertex operator construction of Kac-Moody algebras, Calabi-Yau manifolds as possible ground states of string theory, the sum over Riemann surfaces formulation of string theory, the important role of modular forms and theta functions in conformal field theory, the “Moonshine” meromorphic conformal field theory as a natural representation space for the Monster sporadic simple group. These few examples show sufficiently that it is necessary to learn from each other and to transfer tools and techniques.

In this work we are concerned with the mathematical theories of vertex algebras and Borcherds algebras which were motivated and initiated by developments in physics. After a period of rapid development which has not ended yet we believe that it is perhaps useful to make the rigorous mathematical framework more accessible to physicists. We shall skip the interesting history of the subject since it is presented in great detail in the book [32]. Instead we would like to emphasize that this paper serves various purposes.

First of all we want to provide a pedagogical and self-contained introduction into the area of vertex algebras and related subjects. Although the reader who is familiar with conformal field theory might devote himself to the new formalism more relaxed it could be also quite instructive to learn vertex algebras from scratch. It is clear that in such a treatment only the basics of the theory of vertex algebras can be presented but after reading this review it should be no problem to become familiar with the advanced topics in the mathematics literature.

On the other hand we hope to have compiled a comprehensive “dictionary” which enables a physicist to translate easily formulas and recent results of this mathematically rigorous axiomatic formulation of conformal field theory into his language. Throughout this paper we will stress that vertex algebras establish a solid foundation for the algebraic aspects of conformal field theory.

Of course, vertex algebras constitute a beautiful mathematical theory in their own right connecting many different areas such as the representation theory of the Virasoro algebra and affine Lie algebras, the theory of Riemann surfaces, knot invariants, quantum groups. But we find it especially intriguing for a physicist to see how far one can get by purely formal manipulations as soon as the algebraic structure is extracted from a physical theory.

We also review the present knowledge about Borcherds algebras since up to now physicists have shown surprisingly little interest in this topic though Borcherds algebras should be regarded as the most natural generalization of Kac-Moody algebras. Therefore it is a reasonable hope that they will emerge as new large symmetry algebras in physical models. In this context especially the fake Monster Lie algebra seemingly plays an important role in bosonic string theory.

Finally we have also included some new material. In Section 3.6 we give a natural definition of normal ordered product for vertex operators to interpret the finiteness condition for vertex operator algebras and to exhibit the occurrence of a Gerstenhaber-like algebraic structure. In Section 4.6 we work out in detail the generators, the bracket relations and the Cartan matrix for the fake Monster Lie algebra.

Let us briefly summarize how the paper is organized.

In Section 2 we present the whole machinery of formal calculus which is the cornerstone of the formalism. Many results are similar to those obtained in complex analysis but here we deal solely with formal variables and formal power series. We have collected all necessary formulas

occurring in the relevant literature (especially [32]).

The first half of Section 3 is devoted to the axiomatic setup of vertex algebras and their fundamental properties. We shall emphasize the connection to conformal field theory. The second half is essentially concerned with the analysis and interpretation of the Jacobi identity for vertex algebras which should be seen as the main axiom of the theory. As an outcome we will discuss the notions of locality and duality, the algebra of primary fields of weight one, the cross-bracket algebra, symmetry products. The standard reference is [28].

Vertex algebras associated with even lattices have their origin in toroidal compactifications of bosonic strings. In Section 4 we construct this important class of examples of vertex algebras (cf. [11]). As an easy application we demonstrate how affine Lie algebras arise in this context. Furthermore, the fake Monster Lie algebra [10] which is the first generic example of a Borcherds algebra, is worked out in detail.

After a definition of Borcherds algebras via generators and relations we summarize in Section 5 the basic properties of these generalized Kac-Moody algebras [6], [7]. We also discuss Slansky's investigation [66] of the simplest nontrivial examples of Borcherds algebras and end with a short introduction to the Monster Lie algebra.

In Section 6 we will finally mention the topics which we have not treated in this introductory text and we shall give a brief status report of the areas of current research on the field of vertex algebras.

2 Formal Calculus

A nice exposition of vertex operator formal calculus can be found in [33]. We shall closely follow [32] where the subject is treated thoroughly.

2.1 Notation

In contrast to conformal field theory (see [2], [36] or [57], e.g.), in the vertex algebra approach we use *formal* variables z, z_0, z_1, z_2, \dots . The great advantage of formal calculus is that we perform purely algebraic manipulations instead of bothering about contour integrals, single-valuedness, complex analysis etc.

The objects we will work with are formal power series. For a vector space W , we set

$$\begin{aligned}
 W\{z\} &= \left\{ \sum_{n \in \mathbb{C}} w_n z^n \mid w_n \in W \right\} \\
 W[\![z, z^{-1}]\!] &= \left\{ \sum_{n \in \mathbb{Z}} w_n z^n \mid w_n \in W \right\} \\
 W[\![z]\!] &= \left\{ \sum_{n \in \mathbb{N}} w_n z^n \mid w_n \in W \right\} \\
 W[z, z^{-1}] &= \left\{ \sum_{n \in \mathbb{Z}} w_n z^n \mid w_n \in W, \text{ almost all } w_n = 0 \right\} \quad (\text{Laurent polynomials}) \\
 W[z] &= \left\{ \sum_{n \in \mathbb{N}} w_n z^n \mid w_n \in W, \text{ almost all } w_n = 0 \right\} \quad (\text{polynomials})
 \end{aligned}$$

where "almost all" means "all but finitely many".

Note that these sets are \mathbb{C} -vector spaces under obvious pointwise operations. We can generalize

above spaces in a straightforward way to the case of several commuting formal variables, e.g. $W[[z_1, z_2^{-1}]] = \{\sum_{m,n \in \mathbb{N}} w_{mn} z_1^m z_2^{-n} | w_{mn} \in W\}$. Though $W\{z\}$ may look strange at first sight due to the sum over the complex numbers it is just another way of writing the elements of $W^{\mathbb{C}} \equiv \{f : \mathbb{C} \rightarrow W\}$ the space of W -valued functions over \mathbb{C} .

Since we will often multiply formal series or add up an infinite number of series it is necessary to introduce the notion of algebraic summability.

Let $(x_i)_{i \in I}$ be a family in $\text{End } W$, the vector space of endomorphisms of W (I an index set). We say that $(x_i)_{i \in I}$ is **summable** if for every $w \in W$, $x_i w = 0$ for all but a finite number of $i \in I$. Then the operator $\sum_{i \in I} x_i$ is well-defined. In general an algebraic limit or a product of formal series is defined if and only if the coefficient of *every* monomial in the formal variables in the formal expression is summable.

An example of a nonexistent product is $(\sum_{n \in \mathbb{N}} z^n)(\sum_{m \in \mathbb{N}} z^{-m})$ where even the coefficient of any monomial z^l is not summable (because it would be \mathbb{N} times the identity $1 \equiv \text{id}_W$).

2.2 δ -series

Recall that for $x \in \mathbb{C}$

$$(1-x)^{-1} = \begin{cases} \sum_{k \in \mathbb{N}} x^k & \text{if } |x| < 1 \\ -x^{-1} \sum_{k \in \mathbb{N}} x^{-k} & \text{if } |x| > 1 \end{cases}$$

If we define

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n \in \mathbb{C}[[z, z^{-1}]]$$

then, formally, this is the Laurent expansion of the classical δ -function at $z = 1$. Indeed, $\delta(z)$ enjoys the following fundamental properties:

Proposition 1 :

1. Let $w(z) \in W[[z, z^{-1}]]$, $a \in \mathbb{C}^{\times}$. Then

$$w(z)\delta(az) = w(a^{-1})\delta(az) \tag{1}$$

2. Let $X(z_1, z_2) \in (\text{End } W)[[z_1, z_1^{-1}, z_2, z_2^{-1}]]$ be such that $\lim_{z_1 \rightarrow z_2} X(z_1, z_2)$ exists (algebraically) and let $a \in \mathbb{C}^{\times}$. Then

$$\begin{aligned} X(z_1, z_2)\delta\left(a \frac{z_1}{z_2}\right) &= X(a^{-1}z_2, z_2)\delta\left(a \frac{z_1}{z_2}\right) \\ &= X(z_1, az_1)\delta\left(a \frac{z_1}{z_2}\right) \end{aligned} \tag{2}$$

Proof:

Write $w(z) = \sum_{n \in \mathbb{Z}} w_n z^n$, $X(z_1, z_2) = \sum_{m,n \in \mathbb{Z}} x_{mn} z_1^m z_2^n$, use the definition of $\delta(z)$ and shift summation indices.

Note that $w(z)$ must be a Laurent polynomial to ensure existence of the product with the δ -series. For explicit calculations it is useful to keep in mind that the substitutions in (1) and (2) correspond formally to $az = 1$ and $az_1/z_2 = 1$, respectively. We want to stress

that in analogy with the theory of distributions an expression like $\delta(z)\delta(z)$ does not exist. Moreover, in (1) and (2) integral powers of z , z_1 and z_2 are required so that $z^{1/2}\delta(z) \neq 1^{1/2}\delta(z)$, $z_1^{1/2}z_2^{1/2}\delta(\frac{z_1}{z_2}) \neq z_2\delta(\frac{z_1}{z_2})$, e.g..

Since we can always (formally) differentiate formal power series it is interesting to study the properties of higher derivations of $\delta(z)$. For this purpose we consider a generating function for all the higher derivatives,

$$\delta(z + z_0) \equiv e^{z_0 \frac{d}{dz}} \delta(z) = \sum_{n \in \mathbb{N}} \frac{1}{n!} z_0^n \delta^{(n)}(z) \quad (3)$$

where $(z + z_0)^n$, $n \in \mathbb{Z}$, is to be expanded in nonnegative powers of z_0 . We will come back to this convention later. It turns out that a generalization of the formula $f(x)\delta^{(n)}(x) = (-1)^n f^{(n)}(0)\delta^{(n)}(x)$ for the classical δ -function is valid for the formal series $\delta(z)$.

Proposition 2 :

1. Let $p(z) \in \mathbb{C}[z, z^{-1}]$ and consider the derivation $D = p(z) \frac{d}{dz}$ of $\mathbb{C}[z, z^{-1}]$. Let $w(z) \in W[z, z^{-1}]$, $a \in \mathbb{C}^\times$, $y \in z_0 \mathbb{C}[[z_0]]$. Then

$$w(z)e^{yD}\delta(az) = (e^{-yD}w)(a^{-1})e^{yD}\delta(az) \quad (4)$$

2. Let $p(z_1, z_2) \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$ and consider the derivations $D_i = p(z_1, z_2) \frac{\partial}{\partial z_i}$, $i = 1, 2$, of $\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$. Let $a \in \mathbb{C}^\times$, $y \in z_0 \mathbb{C}[[z_0]]$ and let $X(z_1, z_2) \in (\text{End } W)[[z_1, z_1^{-1}, z_2, z_2^{-1}]]$ be such that $\lim_{z_1 \rightarrow z_2} X(z_1, z_2)$ exists. Then

$$\begin{aligned} X(z_1, z_2)e^{yD_1}\delta\left(a\frac{z_1}{z_2}\right) &= (e^{-yD_1}X)(a^{-1}z_2, z_2)e^{yD_1}\delta\left(a\frac{z_1}{z_2}\right) \\ X(z_1, z_2)e^{yD_2}\delta\left(a\frac{z_1}{z_2}\right) &= (e^{-yD_2}X)(z_1, az_1)e^{yD_2}\delta\left(a\frac{z_1}{z_2}\right) \end{aligned} \quad (5)$$

Proof:

Since y has no constant term e^{yD} is well-defined. We have the Leibniz rule for D , $w(z) \in W[z, z^{-1}]$, $v(z) \in W\{z\}$, $n \geq 0$,

$$D^n(w(z)v(z)) = \sum_{k=0}^n \binom{n}{k} (D^k w(z)) (D^{n-k} v(z))$$

Hence

$$e^{yD}(w(z)v(z)) = (e^{yD}w(z))(e^{yD}v(z))$$

Apply e^{yD} to $(e^{-yD}w(z))\delta(az) = (e^{-yD}w)(a^{-1})\delta(az)$ and invoke above formula to obtain (4). An obvious extension of (4) to two variables together with (2) gives (5).

If we read off the coefficients of y^n for $n \geq 0$ in above formulas we find

$$\begin{aligned} w(z)D^n\delta(az) &= \sum_{k=0}^n (-1)^k \binom{n}{k} (D^k w)(a^{-1})D^{n-k}\delta(az) \\ X(z_1, z_2)D_1^n\delta\left(a\frac{z_1}{z_2}\right) &= \sum_{k=0}^n (-1)^k \binom{n}{k} (D_1^k X)(a^{-1}z_2, z_2)D_1^{n-k}\delta\left(a\frac{z_1}{z_2}\right) \\ X(z_1, z_2)D_2^n\delta\left(a\frac{z_1}{z_2}\right) &= \sum_{k=0}^n (-1)^k \binom{n}{k} (D_2^k X)(z_1, az_1)D_2^{n-k}\delta\left(a\frac{z_1}{z_2}\right) \end{aligned}$$

all expressions existing.

It is no restriction to consider only derivations of the form $D = p(z) \frac{d}{dz}$, $p(z) \in \mathbb{C}[z, z^{-1}]$, since the derivations of $\mathbb{C}[z, z^{-1}]$ are precisely the endomorphisms of that form. To see this let $d \in (\text{End } \mathbb{C})[z, z^{-1}]$ be a derivation. Set $p(z) := d(z)$, so that $D(z) = d(z)$. We have $D(1) = 0 = d(1)$ because of $d(1) = d(1 \cdot 1) = d(1) + d(1)$, and $d(z^{-1}) = -z^{-2}d(z)$ because of $0 = d(1) = d(z \cdot z^{-1}) = d(z)z^{-1} + d(z^{-1})z$. This shows that D and d agree on all powers of z .

2.3 Expansions of zero

Now we want to introduce the tools for formal calculus which correspond to contour integrals and residues for complex variables. Define

$$\begin{aligned}\mathbb{C}(z) = \mathbb{C}(z^{-1}) &= \{p(z)/q(z) | p(z), q(z) \in \mathbb{C}[z], q \neq 0\} \quad (\text{rational functions}) \\ \mathbb{C}((z)) &= \{p(z)/q(z) | p(z), q(z) \in \mathbb{C}[[z]], q \neq 0\} \\ \mathbb{C}((z^{-1})) &= \{p(z^{-1})/q(z^{-1}) | p(z^{-1}), q(z^{-1}) \in \mathbb{C}[[z^{-1}]], q \neq 0\}\end{aligned}$$

Elements of the latter spaces will often be expressed by analytic functions of z and z^{-1} , respectively. They are understood as formal Taylor or Laurent expansions.

Examples:

$$\begin{aligned}(1+z)^a &= \sum_{n \in \mathbb{N}} \binom{a}{n} z^n \in \mathbb{C}[[z]] \\ (1+z^{-1})^a &= \sum_{n \in \mathbb{N}} \binom{a}{n} z^{-n} \in \mathbb{C}[[z^{-1}]]\end{aligned}$$

In the following we will always (though sometimes not explicitly stated) refer to the **binomial convention** which says that *all binomial expressions are to be expanded in nonnegative integral powers of the second variable*. This is the only point in explicit calculations at which one must not be too sloppy.

Example: for $a \in \mathbb{C}$ the following expressions are in general not the same

$$\begin{aligned}\left(\frac{z_1 - z_2}{z_0}\right)^a &= \sum_{n \in \mathbb{N}} \binom{a}{n} (-1)^n z_0^{-a} z_1^{a-n} z_2^n \\ \left(\frac{-z_2 + z_1}{z_0}\right)^a &= \sum_{n \in \mathbb{N}} \binom{a}{n} (-1)^{a-n} z_0^{-a} z_1^n z_2^{a-n}\end{aligned}$$

With the binomial convention we can rewrite the generating function for the derivatives of the δ -series as

$$z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) = z_0^{-1} e^{-z_2 \frac{\partial}{\partial z_1}} \delta \left(\frac{z_1}{z_0} \right)$$

As an important exercise one may prove subsequent identities which will be extremely useful for vertex operator calculus.

Proposition 3 :

1.

$$z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) = z_1^{-1} \delta \left(\frac{z_0 + z_2}{z_1} \right) \quad (6)$$

2.

$$z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) - z_0^{-1} \delta \left(\frac{-z_2 + z_1}{z_0} \right) = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) \quad (7)$$

where all binomial expressions are expanded in nonnegative integral powers of the second variable.

Now let us define two canonical embeddings of the rational functions

$$\begin{aligned} \iota_+ &: \mathbb{C}(z) \hookrightarrow \mathbb{C}((z)) \\ \iota_- &: \mathbb{C}(z^{-1}) \hookrightarrow \mathbb{C}((z^{-1})) \end{aligned}$$

which denote the formal Laurent expansion in z and z^{-1} , respectively.

Example:

$$\begin{aligned} \iota_+(1-z)^{-1} &= (1-z)^{-1} = \sum_{n \in \mathbb{N}} z^n \\ \iota_-(1-z)^{-1} &= \iota_- \left(-z^{-1}(1-z^{-1})^{-1} \right) = -z^{-1} \sum_{n \in \mathbb{N}} z^{-n} \end{aligned}$$

so that

$$\delta(z) = (\iota_+ - \iota_-)(1-z)^{-1}$$

We introduce a linear map Θ by

$$\begin{aligned} \Theta \equiv \Theta_z : \mathbb{C}(z) &\rightarrow \mathbb{C}[[z, z^{-1}]] \\ f &\mapsto \iota_+ f - \iota_- f \end{aligned}$$

We observe that $\ker \Theta = \mathbb{C}[z, z^{-1}]$ i.e. the Laurent polynomials are precisely those formal series for which the expansions in z and z^{-1} , respectively, agree. Moreover, by the partial fraction decomposition of a rational function, we see that the family $\{(1-az)^{-n-1} | n \geq 0, a \in \mathbb{C}^\times\}$ spans a linear complement of $\mathbb{C}[z, z^{-1}]$ in $\mathbb{C}(z)$. The elements of $\text{im } \Theta$ are called **expansions of zero**. The most prominent examples of expansions of zero are given by the δ -series and its derivations.

Proposition 4 :

For $n \in \mathbb{N}$, $a \in \mathbb{C}^\times$

$$\begin{aligned} \frac{1}{n!} \delta^{(n)}(az) &= \Theta \left((1-az)^{-n-1} \right) \\ &= (1-az)^{-n-1} - (-az)^{-n-1} (1-a^{-1}z^{-1})^{-n-1} \end{aligned} \quad (8)$$

i.e.

$$\delta(z + z_0) \equiv e^{z_0 \frac{d}{dz}} \delta(z) = \sum_{n \in \mathbb{N}} \Theta \left((1-z)^{-n-1} \right) z_0^n$$

Proof:

The case $n = 0$ being clear use the fact that Θ commutes with $\frac{d}{dz}$ together with the formula $\frac{1}{n!} \left(\frac{d}{dz} \right)^n (1-z)^{-1} = (1-z)^{-n-1}$ to obtain the case $n > 0$.

Thus the set $\{\delta^{(n)}(az) | n \in \mathbb{N}, a \in \mathbb{C}^\times\}$ is a basis of the space $\text{im}\Theta$ of expansions of zero.

Next we shall generalize the map Θ to the case of two formal variables. Let S denote the set of nonzero linear polynomials in variables z_1 and z_2 ,

$$S = \{az_1 + bz_2 | a, b \in \mathbb{C}, |a| + |b| \neq 0\} \subset \mathbb{C}[z_1, z_2]$$

Consider the subring $\mathbb{C}[z_1, z_2]_S$ of the field of rational functions $\mathbb{C}(z_1, z_2)$ obtained by inverting the product of elements of S . We can write any $f(z_1, z_2) \in \mathbb{C}[z_1, z_2]_S$ in the form

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{z_{i_2}^s \prod_{l=1}^r (a_l z_{i_1} + b_l z_{i_2})}$$

where $g(z_1, z_2) \in \mathbb{C}[z_1, z_2]$, $r, s \in \mathbb{N}$, $a_l \neq 0$ for $l = 1, \dots, r$.

For a permutation $(i_1 \ i_2)$ of $(1 \ 2)$ we define the map

$$\iota_{z_{i_1} z_{i_2}} \equiv \iota_{i_1 i_2} : \mathbb{C}[z_1, z_2]_S \rightarrow \mathbb{C}[[z_1, z_1^{-1}, z_2, z_2^{-1}]]$$

such that each factor $(a_l z_{i_1} + b_l z_{i_2})^{-1}$ in $f \in \mathbb{C}[z_1, z_2]_S$ is expanded in nonnegative integral powers of z_{i_2} . Clearly the maps $\iota_{i_1 i_2}$ are injective.

To obtain "expansions of zero" in the variables z_1, z_2 we set

$$\begin{aligned} \Theta_{i_1 i_2} \equiv \Theta_{z_{i_1} z_{i_2}} : \mathbb{C}[z_1, z_2]_S &\rightarrow \mathbb{C}[[z_1, z_1^{-1}, z_2, z_2^{-1}]] \\ f &\mapsto \iota_{i_1 i_2} f - \iota_{i_2 i_1} f \end{aligned}$$

Then $\ker \Theta_{i_1 i_2} = \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$.

We shall use the following residue notation. For a formal series

$$w(z) = \sum_{n \in \mathbb{C}} w_n z^n \in W\{z\}$$

we write

$$\text{Res}_z [w(z)] = w_{-1}$$

Formal residue enjoys some properties of contour integration:

Proposition 5 :

1. Let $w(z) = \sum_{n \in \mathbb{C}} w_n z^n \in W\{z\}$. For $n \in \mathbb{C}$

$$w_n = \text{Res}_z [z^{-n-1} w(z)] \tag{9}$$

2. (Integration by parts) Let $v(z), w(z) \in W\{z\}$. Then

$$\text{Res}_z \left[v(z) \frac{d}{dz} w(z) \right] = -\text{Res}_z \left[w(z) \frac{d}{dz} v(z) \right] \tag{10}$$

3. (Cauchy theorem)

$$\text{Res}_{z_1 - z_2} [\iota_{z_1, z_1 - z_2} f(z_1, z_2)] = \text{Res}_{z_2} [(\iota_{z_1 z_2} - \iota_{z_2 z_1}) f(z_1, z_2)] \equiv \text{Res}_{z_2} [\Theta_{z_1 z_2} f(z_1, z_2)] \tag{11}$$

for $f(z_1, z_2) = \frac{g(z_1, z_2)}{z_1^r z_2^s (z_1 - z_2)^t}$, $r, s, t \in \mathbb{Z}$, $g(z_1, z_2) \in \mathbb{C}[z_1, z_2]$.

Proof:

1. Clear
2. Use the fact that $\text{Res}_z \left[\frac{d}{dz} u(z) \right] = 0$ for all $u(z) \in W\{z\}$.
3. It is sufficient to consider $h(z_1, z_2) = z_1^l z_2^m (z_1 - z_2)^n$, $l, m, n \in \mathbb{Z}$.

$$\begin{aligned}\iota_{21} h &= \sum_{j \in \mathbb{N}} (-1)^{n-j} \binom{n}{j} z_1^{l+j} z_2^{m+n-j} \\ \iota_{12} h &= \sum_{j \in \mathbb{N}} (-1)^j \binom{n}{j} z_1^{l+n-j} z_2^{m+j} \\ \iota_{1,1-2} h &= \sum_{j \in \mathbb{N}} (-1)^j \binom{m}{j} z_1^{l+m-j} (z_1 - z_2)^{n+j}\end{aligned}$$

This implies

$$\begin{aligned}\text{Res}_{z_2} [\iota_{21} h] &= (-1)^{-m-1} \binom{n}{n+m+1} z_1^{l+m+n+1} \\ \text{Res}_{z_2} [\iota_{12} h] &= (-1)^{-m-1} \binom{n}{-m-1} z_1^{l+m+n+1} \\ \text{Res}_{z_1-z_2} [\iota_{1,1-2} h] &= (-1)^{-n-1} \binom{m}{-n-1} z_1^{l+m+n+1}\end{aligned}$$

i.e. we have to show

$$-(-1)^{m+1} \binom{n}{n+m+1} + (-1)^{m+1} \binom{n}{-m-1} = (-1)^{n+1} \binom{m}{-n-1} \quad \text{for all } m, n \in \mathbb{Z}$$

If $m, n < 0$ then $\binom{n}{n+m+1} = 0$ and above equation holds.

If $n+1 \leq 0 \leq m$ then $\binom{n}{-m-1} = 0$ and above equation holds.

If $0 \leq -m-1 \leq n$ then $\binom{m}{-n-1} = 0$ and above equation holds.

In all other cases the binomial coefficients vanish identically.

Note the wrong sign in the version of Cauchy's theorem given in [34] and the incorrect statement about the property of the δ -series.

We also mention that Cauchy's theorem is equivalent to (7): Just multiply (11) for the special case $f(z_1, z_2) = z_2^m (z_1 - z_2)^n$ with $z_2^{-m-1} z_0^{-n-1}$ and sum over $n, m \in \mathbb{Z}$ to obtain (7).

2.4 Projective change of variables

We have already used exponentials of derivatives like $e^{z_0 \frac{d}{dz}}$ in deriving formulae for the higher derivatives of $\delta(z)$. However, one might also expect $e^{z_0 \frac{d}{dz}}$ to act somehow as a one-parameter group of automorphisms (parametrized by z_0). This turns out to be true in the following sense.

Proposition 6 :

Let $w(z) = \sum_{m \in \mathbb{C}} w_m z^m \in W\{z\}$, $y \in z_0 \mathbb{C}[[z_0]]$ and write $D_n = -z^{n+1} \frac{d}{dz}$, $n \in \mathbb{N}$. Then we have

1. (Translation)

$$e^{-yD_{-1}}w(z) \equiv e^{y\frac{d}{dz}}w(z) = w(z+y) \quad (12)$$

2. (Scaling)

$$(e^y)^{-D_0}w(z) \equiv e^{yz\frac{d}{dz}}w(z) = w(e^y z) \quad (13)$$

3. (Projective change)

$$e^{yD_n}w(z) = w((z^{-n} + ny)^{-1/n}) \quad \text{for } n \neq 0 \quad (14)$$

with binomial convention.

Proof:

1. Write out the expressions as sums
2. Write out the expressions as sums
3. We have $D_n = n\frac{d}{d(z^{-n})}$ for $n \neq 0$. Thus, by (12) ,

$$e^{yD_n} \left(\sum_{m \in \mathbb{C}} w_m (z^{-n})^{-m/n} \right) = \sum_{m \in \mathbb{C}} w_m (z^{-n} + ny)^{-m/n} = w((z^{-n} + ny)^{-1/n})$$

Note that we have already made use of (12) symbolically in (3) and Proposition 4.

For later discussion of meromorphic conformal field theory (see also [37]) it is important to observe that $\{D_{-1}, D_0, D_1\}$ generate a representation of the group of Möbius transformations by

$$z \mapsto \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{\frac{b}{d}\rho(D_{-1})} d^{-2\rho(D_0)} e^{-\frac{c}{d}\rho(D_1)}, \quad ad - bc = 1$$

where the identification is given by

$$\begin{aligned} \rho : \text{span}\{D_{-1}, D_0, D_1\} &\xrightarrow{\cong} \mathfrak{su}(1, 1) \\ D_{-1} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_0 \mapsto \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad D_1 \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

so that

$$e^{z_0\rho(D_{-1})} = \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix}, \quad e^{z_0\rho(D_0)} = \begin{pmatrix} e^{z_0/2} & 0 \\ 0 & e^{-z_0/2} \end{pmatrix}, \quad e^{z_0\rho(D_1)} = \begin{pmatrix} 1 & 0 \\ -z_0 & 1 \end{pmatrix}$$

The full set of D_n 's, however, establishes a representation of the **Witt algebra**,

$$[D_m, D_n] = (m - n)D_{m+n}$$

the central extension of which is the essential ingredient of two-dimensional conformal field theory.

3 Vertex algebras

3.1 Axiomatics of vertex algebras

We shall give a definition of vertex (operator) algebra (cf. [28]) using the notation of [37] which we believe is more accessible to physicists.

Definition 1 :

A vertex algebra is a \mathbb{Z} -graded vector space

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{(n)}$$

equipped with a linear map $\mathcal{V} : \mathcal{F} \rightarrow (\text{End } \mathcal{F})[[z, z^{-1}]]$ which assigns to each state $\psi \in \mathcal{F}$ a **vertex operator** $\mathcal{V}(\psi, z)$, and the vertex operators satisfy the following axioms:

1. **(Regularity)** If $\psi, \varphi \in \mathcal{F}$ then

$$\text{Res}_z [z^n \mathcal{V}(\psi, z) \varphi] = 0 \quad \text{for } n \text{ sufficiently large} \quad (15)$$

and n depending on ψ and φ

2. **(Vacuum)** There is a preferred state $\mathbf{1} \in \mathcal{F}$, called the **vacuum**, satisfying

$$\mathcal{V}(\mathbf{1}, z) = \text{id}_{\mathcal{F}} \quad (16)$$

3. **(Injectivity)** There is a one-to-one correspondence between states and vertex operators,

$$\mathcal{V}(\psi, z) = 0 \iff \psi = 0 \quad (17)$$

4. **(Conformal vector)** There is a preferred state $\omega \in \mathcal{F}$, called the **conformal vector**, such that its vertex operator

$$\mathcal{V}(\omega, z) = \sum_{n \in \mathbb{Z}} L_{(n)} z^{-n-2} \quad (18)$$

(a) gives the **Virasoro algebra** with some central charge $c \in \mathbb{C}$,

$$[L_{(m)}, L_{(n)}] = (m - n)L_{(m+n)} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad (19)$$

(b) provides a **translation generator**, $L_{(-1)}$,

$$\mathcal{V}(L_{(-1)}\psi, z) = \frac{d}{dz} \mathcal{V}(\psi, z) \quad \text{for every } \psi \in \mathcal{F} \quad (20)$$

(c) gives the **grading** of \mathcal{F} via the eigenvalues of $L_{(0)}$,

$$L_{(0)}\psi = n\psi \equiv \Delta_\psi \psi \quad \text{for every } \psi \in \mathcal{F}_{(n)}, n \in \mathbb{Z} \quad (21)$$

the eigenvalue Δ_ψ is called the **(conformal) weight** of ψ .

5. (**Jacobi identity**) For every $\psi, \varphi \in \mathcal{F}$,

$$\begin{aligned} z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) \mathcal{V}(\psi, z_1) \mathcal{V}(\varphi, z_2) - z_0^{-1} \delta \left(\frac{-z_2 + z_1}{z_0} \right) \mathcal{V}(\varphi, z_2) \mathcal{V}(\psi, z_1) \\ = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) \mathcal{V}(\psi, z_0) \varphi, z_2 \end{aligned}$$

where binomial expressions have to be expanded in nonnegative integral powers of the second variable.

We denote the vertex algebra just defined by $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$.

We may think of \mathcal{F} as the space of finite occupation number states in a Fock space so that \mathcal{F} is a dense subspace of the Hilbert space \mathcal{H} of states. The regularity axiom states that, given $\psi, \varphi \in \mathcal{F}$, there is always a high enough power z^n such that $z^n \mathcal{V}(\psi, z) \varphi$ is (at “ $z = 0$ ”) a regular formal series. In other words, the regularity axiom ensures that any $\mathcal{V}(\psi, z) \varphi$ contains only a *finite* number of singular expressions. In terms of creation and annihilation operators it reflects the fact that any state φ is killed by a finite but large enough number of annihilation operators contained in (the normal ordered expression) ψ_n . We also mention that in physical applications the vertex operator of the conformal vector corresponds to the stress–energy tensor of the field theory.

Definition 2 :

A vertex operator algebra is a vertex algebra with the additional assumptions that

1. the spectrum of $L_{(0)}$ is bounded below
2. the eigenspaces $\mathcal{F}_{(n)}$ of $L_{(0)}$ are finite-dimensional.

The first condition is an immediate consequence of a physical postulate. As we will see $L_{(0)}$ generates scale transformations. Recalling that the variable z in conformal field theory has its origin in e^{t+ix} (cf. [36]) one finds that $L_{(0)}$ corresponds to time translations. Thus it may be identified with the energy which should be bounded below in any sensible quantum field theory. In fact, vertex operator algebras can be regarded as a rigorous mathematical definition of chiral algebras in physics [57]. Then the formal variable z can be thought of as a local complex coordinate. The vertex operators $\mathcal{V}(\psi, z)$ correspond to holomorphic chiral fields i.e. they can be viewed as operator-valued distributions on a local coordinate chart of a Riemann surface. In this context the three terms terms of the Jacobi identity are geometrically interpreted as the three ways of cutting the four–punctured Riemann sphere into two three–punctured spheres. Alternatively, one might regard the Jacobi identity as a precise statement of the Ward identities on the three–punctured Riemann sphere. [34],[74]

Since vertex operators are operator valued formal Laurent series we can give an alternative formulation (see [10], e.g.) of the axioms of a vertex algebra using the mode expansion

$$\mathcal{V}(\psi, z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1} \tag{22}$$

One has

1. (Regularity)*

$$\psi_n \varphi = 0 \quad \text{for } n \text{ sufficiently large} \quad (23)$$

2. (Vacuum)*

$$\mathbf{1}_n \psi = \delta_{n+1,0} \psi \quad (24)$$

3. (Injectivity)*

$$\psi_n = 0 \quad \forall n \in \mathbb{Z} \quad \iff \quad \psi = 0 \quad (25)$$

4. (Conformal vector)*

$$\omega_{n+1} = L_{(n)} \quad (26)$$

5. (Jacobi identity)*

$$\sum_{i \geq 0} (-1)^i \binom{l}{i} (\psi_{l+m-i}(\varphi_{n+i} \xi) - (-1)^l \varphi_{l+n-i}(\psi_{m+i} \xi)) = \sum_{i \geq 0} \binom{m}{i} (\psi_{l+i} \varphi)_{m+n-i} \xi \quad (27)$$

for all $\psi, \varphi, \xi \in \mathcal{F}$, $l, m, n \in \mathbb{Z}$.

To see the equivalence of the two formulations of the Jacobi identity evaluate

$\text{Res}_{z_2} \left[\text{Res}_{z_1} \left[\text{Res}_{z_0} \left[z_2^n z_1^m z_0^l (\text{Jacobi identity}) \right] \right] \right]$. As an intermediate result one finds, by using (6), yet another version of the Jacobi identity which occurs in the literature [74],[34]:

$$\begin{aligned} \text{Res}_{z_1} \left[\mathcal{V}(\psi, z_1) \mathcal{V}(\varphi, z_2) \iota_{12} \left((z_1 - z_2)^l z_1^m \right) - \mathcal{V}(\varphi, z_2) \mathcal{V}(\psi, z_1) \iota_{21} \left((z_1 - z_2)^l z_1^m \right) \right] \\ = \text{Res}_{z_0} \left[\mathcal{V}(\mathcal{V}(\psi, z_0) \varphi, z_2) \iota_{20} \left(z_0^l (z_2 + z_0)^m \right) \right] \end{aligned}$$

As one might suspect the Jacobi identity contains most information of a vertex algebra. In fact we will see that one can derive from it important properties such as locality and duality in conformal field theory (cf. [37]). If we put $l = m = n = 0$ in (27) we get $\psi_0(\varphi_0 \xi) - \varphi_0(\psi_0 \xi) = (\psi_0 \varphi)_0 \xi$. Later we will define an antisymmetric product on the subspace $\mathcal{F}/L_{(-1)}\mathcal{F}$ of \mathcal{F} by $[\psi, \varphi] := \psi_0 \varphi$. Then above formula indeed establishes the classical Jacobi identity for Lie algebras. On the other hand, choosing $\psi = \varphi = \mathbf{1}$ in the Jacobi identity for vertex algebras and using the vacuum axiom we recover (7) and thus the Cauchy theorem (11). Hence the Jacobi identity for vertex algebras may be regarded as a combination of the classical Jacobi identity for Lie algebras and the Cauchy residue formula for meromorphic functions.

In what follows we will frequently make use of two important formulas which are the special cases $m = 0$ and $l = 0$, respectively, of (27) :

(Associativity formula)

$$(\psi_l \varphi)_n = \sum_{i \geq 0} (-1)^i \binom{l}{i} (\psi_{l-i} \varphi_{n+i} - (-1)^l \varphi_{l+n-i} \psi_i) \quad (28)$$

(Commutator formula)

$$[\psi_m, \varphi_n] = \sum_{i \geq 0} \binom{m}{i} (\psi_i \varphi)_{m+n-i} \quad (29)$$

for all $\psi, \varphi \in \mathcal{F}$, $l, m, n \in \mathbb{Z}$.

From the commutator formula we can immediately infer the interesting result that in any vertex algebra the zero mode operators, ψ_0 , act as derivations on the products $\varphi_n \chi$, i.e.,

$$\psi_0(\varphi_n \chi) = (\psi_0 \varphi)_n \chi + \varphi_n(\psi_0 \chi) \quad (30)$$

3.2 Basic properties of vertex algebras

To become familiar with the definition let us derive some interesting properties of vertex algebras (In [74] some consequences of the definition are stated incorrectly). Iterating (20) and using translation (12) we find that $L_{(-1)}$ indeed generates translations,

$$\mathcal{V}\left(e^{z_0 L_{(-1)}} \psi, z\right) = \mathcal{V}(\psi, z + z_0) \quad (31)$$

Moreover, the vacuum is translation invariant because (20) for $\psi = \mathbf{1}$ together with the vacuum axiom (16) and injectivity (17) gives

$$L_{(-1)} \mathbf{1} = 0 \quad (32)$$

Take $\text{Res}_{z_0} [\text{Res}_{z_1} [z_1^n (\text{Jacobi identity})]]$,

$$[\psi_n, \mathcal{V}(\varphi, z)] = \sum_{i \geq 0} \binom{n}{i} \mathcal{V}(\psi_i \varphi, z) z^{n-i}$$

In the special case $\psi = \omega$ we obtain

$$[L_{(n)}, \mathcal{V}(\varphi, z)] = \sum_{i \geq -1} \binom{n+1}{i+1} \mathcal{V}(L_{(i)} \varphi, z) z^{n-i}$$

in particular,

$$[L_{(-1)}, \mathcal{V}(\varphi, z)] = \frac{d}{dz} \mathcal{V}(\varphi, z) \quad (33)$$

$$[L_{(0)}, \mathcal{V}(\varphi, z)] = \left(z \frac{d}{dz} + \Delta_\varphi \right) \mathcal{V}(\varphi, z) \quad \text{if } \varphi \in \mathcal{F}_{(\Delta_\varphi)} \quad (34)$$

Using the well-known formula $e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}_A)^n B \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A, [A, \dots [A, B]] \dots]}_n$ these

equations give, respectively,

(Translation property)

$$e^{y L_{(-1)}} \mathcal{V}(\varphi, z) e^{-y L_{(-1)}} = \mathcal{V}(\varphi, z + y) \quad (35)$$

(Scaling property)

$$e^{y L_{(0)}} \mathcal{V}(\varphi, z) e^{-y L_{(0)}} = e^{y \Delta_\varphi} \mathcal{V}(\varphi, e^y z) \quad \text{if } \varphi \in \mathcal{F}_{(\Delta_\varphi)} \quad (36)$$

for every $y \in z_0 \mathbb{C}[[z_0]]$ by Proposition 3.

Thus $L_{(0)}$ generates scale transformations. Note that (34) also implies

$$\varphi_n \mathcal{F}_{(m)} \subset \mathcal{F}_{(\Delta_\varphi + m - n - 1)} \quad \text{if } \varphi \in \mathcal{F}_{(\Delta_\varphi)} \quad (37)$$

which means that the operator φ_n shifts the grading by $\Delta_\varphi - n - 1$, i.e. it can be assigned “degree” $\Delta_\varphi - n - 1$. In view of this relation the reader might wonder again why we use subscripts in round brackets for the grading of \mathcal{F} and for the Virasoro generators in contrast to the naked subscripts occurring in the mode expansion (22) of a vertex operator. This possibly causes some confusion but stems from the fact that we employ two different mode expansions. In conformal field theory we are acquainted with the expansion

$$\psi(z) \equiv \mathcal{V}(\psi, z) = \sum_{n \in \mathbb{Z}} \psi_{(n)} z^{-n - \Delta_\psi} \quad (38)$$

which depends on the conformal weight of the field $\psi(z)$. To exhibit explicitly the Virasoro algebra in the definition of a vertex algebra we used this expansion for the vertex operator associated with the conformal vector (stress-energy tensor!) in (18). It is quite easy to convert results obtained in one expansion into the other formalism, namely, simply by shifting the grading:

$$\begin{aligned}\psi_n &\equiv \psi_{(n+1-\Delta_\psi)} \\ \psi_{(n)} &\equiv \psi_{n-1+\Delta_\psi}\end{aligned}$$

for any homogeneous element $\psi \in \mathcal{F}$. For example we can rewrite (37) as

$$\varphi_{(n)}\mathcal{F}_{(m)} \subset \mathcal{F}_{(m-n)}$$

so that $\varphi_{(n)}$ always has “degree” $-n$ irrespective of φ . In so far the mode expansion (38) is therefore the more natural one because it respects the grading of \mathcal{F} . On the other hand for formal calculus it is more useful to stick to an expansion which does not refer to the conformal weight of a state. Hence we shall almost everywhere in the formulae assume the mode expansion (22).

Let us exploit the fact that the Jacobi identity is obviously invariant under $(\psi, z_1, z_0) \leftrightarrow (\varphi, z_2, -z_0)$:

$$\begin{aligned}z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2) &= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\mathcal{V}(\mathcal{V}(\varphi, -z_0)\psi, z_1) \quad \text{by symmetry} \\ &= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\mathcal{V}(\mathcal{V}(\varphi, -z_0)\psi, z_2+z_0) \quad \text{by (5)} \\ &= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{V}(\mathcal{V}(\varphi, -z_0)\psi, z_2+z_0) \quad \text{by (6)}\end{aligned}$$

Taking $\text{Res}_{z_1}[\dots]$ we get

$$\begin{aligned}\mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2) &= \mathcal{V}(\mathcal{V}(\varphi, -z_0)\psi, z_2+z_0) \\ &= \mathcal{V}\left(e^{z_0\mathbf{L}_{(-1)}}\mathcal{V}(\varphi, -z_0)\psi, z_2\right) \quad \text{by (31)}\end{aligned}$$

Injectivity (17) finally yields

(Skew-symmetry)

$$\mathcal{V}(\psi, z_0)\varphi = e^{z_0\mathbf{L}_{(-1)}}\mathcal{V}(\varphi, -z_0)\psi \quad (39)$$

or, in components,

$$\psi_n\varphi = -(-1)^n\varphi_n\psi + \sum_{i \geq 1} \frac{1}{i!}(-1)^{i+n+1} \left(\mathbf{L}_{(-1)}\right)^i (\varphi_{n+i}\psi) \quad (40)$$

In particular, we observe that the vertex operator $\mathcal{V}(\psi, z)$ “creates” the state $\psi \in \mathcal{F}$ when applied to the vacuum:

$$\mathcal{V}(\psi, z)\mathbf{1} = e^{z\mathbf{L}_{(-1)}}\psi \quad (41)$$

by (16). In components,

$$\psi_n\mathbf{1} = \begin{cases} 0 & \text{for } n \geq 0 \\ \psi & \text{for } n = -1 \\ \frac{1}{(-n-1)!} \left(\mathbf{L}_{(-1)}\right)^{-n-1} \psi & \text{for } n \leq -2 \end{cases} \quad (42)$$

Hence the vacuum satisfies

$$L_{(n)}\mathbf{1} = 0 \quad \forall n \geq -1 \quad (43)$$

We shall denote by $\mathcal{P}_{(\Delta)}$ the space of **(conformal) highest weight vectors or primary states** satisfying

$$\begin{aligned} L_{(0)}\psi &= \Delta\psi \quad \text{i.e. } \psi \in \mathcal{F}_{(\Delta)} \\ L_{(n)}\psi &= 0 \quad \forall n > 0 \end{aligned} \quad (44)$$

Thus in any vertex algebra the vacuum is a primary state of weight zero. We immediately find for $\psi \in \mathcal{P}_{(\Delta)}$

$$[L_{(n)}, \mathcal{V}(\psi, z)] = z^n \left\{ z \frac{d}{dz} + (n+1)\Delta \right\} \mathcal{V}(\psi, z) \quad \forall n \in \mathbb{Z} \quad (45)$$

or

$$[L_{(n)}, \psi_m] = \{(\Delta - 1)(n+1) - m\} \psi_{m+n} \quad \forall m, n \in \mathbb{Z} \quad (46)$$

i.e. $\mathcal{V}(\psi, z)$ is a so called **(conformal) primary field** of weight Δ . We can rewrite (45) as

$$[L_{(n)}, z^{\Delta(n+1)} \mathcal{V}(\psi, z)] = z^{n+1} \frac{d}{dz} \left\{ z^{\Delta(n+1)} \mathcal{V}(\psi, z) \right\}$$

so that, by (14),

$$e^{yL_{(n)}} \mathcal{V}(\psi, z) e^{-yL_{(n)}} = \left(\frac{\partial z_1}{\partial z} \right)^\Delta \mathcal{V}(\psi, z_1) \quad \forall n \neq 0 \quad (47)$$

for every $y \in z_0 \mathbb{C}[[z_0]]$ where $z_1 = (z^{-n} - ny)^{-1/n} = z(1 - nyz^n)^{-1/n}$

The operators $\{L_{(-1)}, L_{(0)}, L_{(1)}\}$ satisfy the $\mathfrak{su}(1, 1)$ Lie algebra

$$[L_{(0)}, L_{(1)}] = -L_{(1)}, \quad [L_{(0)}, L_{(-1)}] = L_{(-1)}, \quad [L_{(1)}, L_{(-1)}] = 2L_{(0)}$$

Hence we have established the following Möbius transformation properties of the vertex operators (see also [37]):

If $\psi \in \mathcal{F}$ is a **quasi-primary state** of weight Δ , i.e. ψ satisfies $L_{(n)}\psi = \delta_{n,0}\Delta\psi$, $n = 0, 1$, then

$$D_\gamma \mathcal{V}(\psi, z) D_\gamma^{-1} = \left(\frac{d\gamma(z)}{dz} \right)^\Delta \mathcal{V}(\psi, \gamma(z))$$

where

$$\gamma(z) = \frac{az + b}{cz + d}, \quad D_\gamma = e^{\frac{b}{d}L_{(-1)}} \left(\frac{\sqrt{ad - bc}}{d} \right)^{2L_{(0)}} e^{-\frac{c}{d}L_{(1)}} \quad \text{for } a, b, c, d \in z_0 \mathbb{C}[[z_0]]$$

(cf. end of Subsection 2.4)

Now equation (43) tells us that the vacuum vector is $SU(1,1)$ invariant and the question arises whether the state $\mathbf{1}$ is uniquely (up to scalar multiples) characterized by this property. In general the answer is no but $SU(1,1)$ invariant states come quite close to the properties of the vacuum. To see this suppose that $\kappa \in \mathcal{F}$ satisfies $L_{(n)}\kappa = 0$, $n = 0, \pm 1$. Then, by injectivity and translation, $L_{(-1)}\kappa = 0$ is equivalent to $\kappa_n = \delta_{n+1,0}\kappa_{-1}$ in agreement with (24). However, as κ_{-1} regards the associativity formula (28) and the commutator formula (29) yield $(\kappa_{-1}\varphi)_n = \kappa_{-1}\varphi_n = \varphi_n\kappa_{-1} \forall \varphi \in \mathcal{F}, n \in \mathbb{Z}$. For general vertex algebras this does *not* force κ_{-1} to be a scalar multiple of the identity but rather states that κ_{-1} may be regarded as a Casimir

operator of the vertex algebra. In the case of a *simple* vertex algebra (i.e. the vertex algebra constitutes an irreducible module for itself, cf. [28]) we can apply Schur's lemma [35] to infer that ω_{-1} is indeed a scalar multiple of the identity.

Finally, using the Virasoro algebra (19) and (42) we find

$$\begin{aligned}\omega_l\omega &= \omega_l(\omega_{-1}\mathbf{1}) \\ &= [\omega_l, \omega_{-1}]\mathbf{1} + \omega_{-1}(\omega_l\mathbf{1}) \\ &= (l+1)\omega_{l-2}\mathbf{1} + \frac{c}{2}\delta_{l,3}\mathbf{1} + \omega_{-1}(\omega_l\mathbf{1})\end{aligned}$$

i.e. (cf. [10])

$$\begin{aligned}\omega_1\omega &= 2\omega \\ \omega_2\omega &= 0 \\ \omega_3\omega &= \frac{c}{2} \\ \omega_l\omega &= 0 \quad \text{for } l > 3\end{aligned}$$

In particular, ω is a quasi-primary state of conformal weight two. We want to mention that ω is characterized by above relations together with (21) and $\omega_0\psi = \psi_{-2}\mathbf{1}$ for $\psi \in \mathcal{F}$.

3.3 Locality and duality for vertex algebras

To complete the relation between vertex algebras and conformal field theory we consider matrix elements of products of vertex operators. Define the **restricted dual** of \mathcal{F} ,

$$\mathcal{F}' \equiv \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{(n)}^*$$

the direct sum of the dual spaces of the homogeneous subspaces $\mathcal{F}_{(n)}$, i.e. the space of linear functionals on the vertex algebra \mathcal{F} vanishing on all but finitely many $\mathcal{F}_{(n)}$. We shall use $\langle _, _ \rangle$ for the natural pairing between \mathcal{F} and \mathcal{F}' . From the regularity axiom and (37) it is clear that any matrix element of a vertex operator is a Laurent polynomial in z ,

$$\langle \chi^* | \mathcal{V}(\psi, z)\varphi \rangle \in \mathbb{C}[z, z^{-1}] \quad \text{for all } \chi^* \in \mathcal{F}', \psi, \varphi \in \mathcal{F} \quad (48)$$

and in this sense these 3-point correlation functions may be regarded as meromorphic functions of z . Of course, we identify formally χ^* with an “out-state” inserted at $z = \infty$ and φ with an “in-state” inserted at $z = 0$. We have the following important theorem due to [32].

Theorem 1 :

1. **(Locality \equiv rationality of products & commutativity)**

For $\chi^* \in \mathcal{F}'$, $\psi, \varphi, \xi \in \mathcal{F}$, the formal series $\langle \chi^* | \mathcal{V}(\psi, z_1)\mathcal{V}(\varphi, z_2)\xi \rangle$ which involves only finitely many negative powers of z_2 and only finitely many positive powers of z_1 , lies in the image of the map ι_{12} :

$$\langle \chi^* | \mathcal{V}(\psi, z_1)\mathcal{V}(\varphi, z_2)\xi \rangle = \iota_{12}f(z_1, z_2) \quad (49)$$

where the (uniquely determined) element $f \in \mathbb{C}[z_1, z_2]_S$ is of the form

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{z_1^r z_2^s (z_1 - z_2)^t}$$

for some polynomial $g(z_1, z_2) \in \mathbb{C}[z_1, z_2]$ and $r, s, t \in \mathbb{Z}$.

We also have

$$\langle \chi^* | \mathcal{V}(\varphi, z_2) \mathcal{V}(\psi, z_1) \xi \rangle = \iota_{21} f(z_1, z_2) \quad (50)$$

i.e. $\mathcal{V}(\psi, z_1) \mathcal{V}(\varphi, z_2)$ agrees with $\mathcal{V}(\varphi, z_2) \mathcal{V}(\psi, z_1)$ as operator-valued rational functions.

2. **(Duality \equiv rationality of iterates & associativity)**

For $\chi^* \in \mathcal{F}'$, $\psi, \varphi, \xi \in \mathcal{F}$, the formal series $\langle \chi^* | \mathcal{V}(\mathcal{V}(\psi, z_0) \varphi, z_2) \xi \rangle$ which involves only finitely many negative powers of z_0 and only finitely many positive powers of z_2 , lies in the image of the map ι_{20} :

$$\langle \chi^* | \mathcal{V}(\mathcal{V}(\psi, z_0) \varphi, z_2) \xi \rangle = \iota_{20} f(z_0 + z_2, z_2) \quad (51)$$

with the same f as above,

$$\langle \chi^* | \mathcal{V}(\psi, z_0 + z_2) \mathcal{V}(\varphi, z_2) \xi \rangle = \iota_{02} f(z_0 + z_2, z_2) \quad (52)$$

i.e. $\mathcal{V}(\psi, z_1) \mathcal{V}(\varphi, z_2)$ agrees with $\mathcal{V}(\mathcal{V}(\psi, z_1 - z_2) \varphi, z_2)$ as operator-valued rational functions, where the right-hand expression is to be expanded as a Laurent series in $z_1 - z_2$.

Proof (taken from [32],[28]):

1. Using (7) we can rewrite the Jacobi identity in the form

$$\begin{aligned} & z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) \mathcal{V}(\psi, z_1) \mathcal{V}(\varphi, z_2) - z_0^{-1} \delta \left(\frac{-z_2 + z_1}{z_0} \right) \mathcal{V}(\varphi, z_2) \mathcal{V}(\psi, z_1) \\ &= \mathcal{V} \left(\left(z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) \mathcal{V}(\psi, z_1 - z_2) - z_0^{-1} \delta \left(\frac{-z_2 + z_1}{z_0} \right) \mathcal{V}(\psi, -z_2 + z_1) \right) \varphi, z_2 \right) \end{aligned}$$

Taking $\text{Res}_{z_0} [\dots]$ leads to the commutator formula

$$[\mathcal{V}(\psi, z_1), \mathcal{V}(\varphi, z_2)] = \mathcal{V}((\mathcal{V}(\psi, z_1 - z_2) - \mathcal{V}(\psi, -z_2 + z_1)) \varphi, z_2)$$

By (48), the matrix element $\langle \chi^* | \dots \xi \rangle$ of the right-hand side is clearly an expansion of zero in the variables z_1, z_2 of the following form:

$$\langle \chi^* | \mathcal{V}((\mathcal{V}(\psi, z_1 - z_2) - \mathcal{V}(\psi, -z_2 + z_1)) \varphi, z_2) \xi \rangle = \Theta_{12} \left(\frac{g(z_1, z_2)}{z_2^s (z_1 - z_2)^t} \right)$$

with $g(z_1, z_2), s, t$ as stated above. Thus

$$\langle \chi^* | \mathcal{V}(\psi, z_1) \mathcal{V}(\varphi, z_2) \xi \rangle - \iota_{12} \left(\frac{g(z_1, z_2)}{z_2^s (z_1 - z_2)^t} \right) = \langle \chi^* | \mathcal{V}(\varphi, z_2) \mathcal{V}(\psi, z_1) \xi \rangle - \iota_{21} \left(\frac{g(z_1, z_2)}{z_2^s (z_1 - z_2)^t} \right)$$

But the left-hand side involves only finitely many positive powers of z_1 , by (37), and the right-hand side involves only finitely many negative powers of z_1 , by the regularity axiom. If we further take into account that, by (48), the coefficient of each power of z_1 on either side is a Laurent polynomial in z_2 then

$$f(z_1, z_2) := \frac{g(z_1, z_2)}{z_2^s (z_1 - z_2)^t} + h(z_1, z_2)$$

for some $h(z_1, z_2) \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$ satisfies the desired conditions.

2. Using (6) and (5) we can rewrite the Jacobi identity in the form

$$\begin{aligned} z_1^{-1} \delta \left(\frac{z_0 + z_2}{z_1} \right) \mathcal{V}(\psi, z_0 + z_2) \mathcal{V}(\varphi, z_2) - z_0^{-1} \delta \left(\frac{-z_2 + z_1}{z_0} \right) \mathcal{V}(\varphi, z_2) \mathcal{V}(\psi, z_1) \\ = z_1^{-1} \delta \left(\frac{z_2 + z_0}{z_1} \right) \mathcal{V}(\mathcal{V}(\psi, z_0) \varphi, z_2) \end{aligned}$$

Thus

$$\begin{aligned} & z_1^{-1} \delta \left(\frac{z_2 + z_0}{z_1} \right) \mathcal{V}(\mathcal{V}(\psi, z_0) \varphi, z_2) - z_1^{-1} \delta \left(\frac{z_0 + z_2}{z_1} \right) \mathcal{V}(\psi, z_0 + z_2) \mathcal{V}(\varphi, z_2) \\ &= \left\{ z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) - z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) \right\} \mathcal{V}(\varphi, z_2) \mathcal{V}(\psi, z_1) \quad \text{by (7)} \\ &= \mathcal{V}(\varphi, z_2) \left\{ z_1^{-1} \delta \left(\frac{z_2 + z_0}{z_1} \right) \mathcal{V}(\psi, z_2 + z_0) - z_1^{-1} \delta \left(\frac{z_0 + z_2}{z_1} \right) \mathcal{V}(\psi, z_0 + z_2) \right\} \quad \text{by (6), (5)} \end{aligned}$$

Taking $\text{Res}_{z_1} [\dots]$ leads to

$$\mathcal{V}(\mathcal{V}(\psi, z_0) \varphi, z_2) - \mathcal{V}(\psi, z_0 + z_2) \mathcal{V}(\varphi, z_2) = \mathcal{V}(\varphi, z_2) \{ \mathcal{V}(\psi, z_2 + z_0) - \mathcal{V}(\psi, z_0 + z_2) \}$$

We use this formula in place of the commutator formula and apply the same arguments as in part one to obtain the desired result. To get the last statement put formally $z_0 = z_1 - z_2$.

The first part of Theorem 1 in particular states that these matrix elements may be viewed as meromorphic functions of the formal variables. Thus vertex algebras can be seen as a rigorous formulation of meromorphic conformal field theories. Note that the second part of the theorem should be interpreted as “duality (crossing symmetry) of the 4-point correlation function on the Riemann sphere”. It establishes a precise formulation of an operator product expansion in two-dimensional conformal field theory [2],[36],[37] in the sense that $\mathcal{V}(\psi, z_1) \mathcal{V}(\varphi, z_2)$ agrees with $\sum_{n \in \mathbb{Z}} (z_1 - z_2)^{-n-1} \mathcal{V}(\psi_n \varphi, z_2)$ as operator valued rational functions. The theorem then also ensures that this operator product expansion involves only finitely many singular (at “ $z_1 = z_2$ ”) terms. It is worth mentioning that Theorem 1 contains the full information about the Jacobi identity, i.e. one can derive the latter starting from the principles of locality and duality. Even more is true. Using products of three vertex operators, duality follows from locality, (33), (34) and the axioms for vertex algebras except for the Jacobi identity. In particular, in the definition of a vertex algebra the Jacobi identity may be replaced by the principle of locality, (33) and (34). Proofs of these statements can be found in [28] where also the generalization of above notion of duality to arbitrary n -point functions is presented.

If we regard our formal variables as *complex* variables then the formal expansions of rational functions that we have been discussing converge in suitable domains. The matrix elements in (49) and (50) converge to a common rational function in the disjoint domains $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively. The matrix elements in (51) and (52) for $z_0 = z_1 - z_2$ converge to a common rational function in the domains $|z_2| > |z_1 - z_2| > 0$ and $|z_1| > |z_2| > 0$, respectively, and in the common domain $|z_1| > |z_2| > |z_1 - z_2| > 0$ these two series converge to the common function.

Finally, we would like to discuss the space \mathcal{F}' and the pairing $\langle \cdot | \cdot \rangle$ in more detail. We can assign to each state $\psi \in \mathcal{F}$ a **contragredient vertex operator** $\mathcal{V}^*(\psi, z) = \sum \psi_n^* z^{-n-1} \in (\text{End } \mathcal{F}')[[z, z^{-1}]]$ by the condition

$$\langle \mathcal{V}^*(\psi, z) \chi^* | \varphi \rangle = \langle \chi^* | \mathcal{V} \left(e^{zL_{(1)}} (-z^{-2})^{L_{(0)}} \psi, z^{-1} \right) \varphi \rangle$$

It is crucial to observe that the formal sum on the right hand side in general does not exist (check!) unless $(L_{(1)})^n \psi = 0$ for n large enough which is assured if the spectrum of $L_{(0)}$ is bounded below. For the remainder of this subsection we will therefore assume that $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ is a vertex operator algebra. Without giving the precise definition of a module for a vertex operator algebra we just state that with this definition $(\mathcal{F}', \mathcal{V}^*)$ becomes a $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ -module (For a proof and the relevant definitions see [28]). If we furthermore have a grading-preserving linear isomorphism $F : \mathcal{F} \rightarrow \mathcal{F}'$ then this amounts to choosing a nondegenerate bilinear form $(_, _)$ on \mathcal{F} as $(\chi, \varphi) := \langle F(\chi) | \varphi \rangle$ with the **adjoint vertex operator** defined by

$$\mathcal{V}^\dagger(\psi, z) = \sum_{n \in \mathbb{Z}} \psi_n^\dagger z^{-n-1} := \mathcal{V}\left(e^{zL_{(1)}}(-z^{-2})^{L_{(0)}}\psi, z^{-1}\right) \in (\text{End } \mathcal{F})[[z, z^{-1}]] \quad (53)$$

such that $(\mathcal{V}(\psi, z)\chi, \varphi) = (\chi, \mathcal{V}^\dagger(\psi, z)\varphi)$.

This definition of an adjoint vertex operator is quite close to the one familiar to physicists as one immediately sees by calculating explicitly the adjoint vertex operator associated with a quasi-primary state ψ .

$$\begin{aligned} \mathcal{V}^\dagger(\psi, z) &= (-1)^{\Delta_\psi} z^{-2\Delta_\psi} \mathcal{V}(\psi, z^{-1}) \quad \text{since } \psi \text{ quasi-primary} \\ &= (-1)^{\Delta_\psi} \sum_{n \in \mathbb{Z}} \psi_n z^{n+1-2\Delta_\psi} \end{aligned}$$

i.e.

$$\psi_n^\dagger = (-1)^{\Delta_\psi} \psi_{-n+2\Delta_\psi-2} \quad \forall n \in \mathbb{Z} \quad (54)$$

With the shifted grading $\psi_{(n)} \equiv \psi_{n+\Delta_\psi-1}$ this reads

$$\psi_{(n)}^\dagger = (-1)^{\Delta_\psi} \psi_{(-n)} \quad \forall n \in \mathbb{Z}$$

In particular, we observe that the vacuum vertex operator (identity) is selfadjoint and that the Virasoro generators satisfy the well-known relation $L_{(n)}^\dagger = L_{(-n)}$ or, in terms of the “stress energy tensor”, $\mathcal{V}^\dagger(\boldsymbol{\omega}, z) = \frac{1}{z^4} \mathcal{V}(\boldsymbol{\omega}, z^{-1})$ [2]. Hence we obtain for any two homogeneous elements $\chi \in \mathcal{F}_{(m)}$, $\varphi \in \mathcal{F}_{(n)}$,

$$(m-n)(\chi, \varphi) = (L_{(0)}\chi, \varphi) - (\chi, L_{(0)}\varphi) = 0$$

i.e. the homogeneous subspaces $\mathcal{F}_{(n)}$, $n \in \mathbb{Z}$, are orthogonal to each other with respect to this bilinear form,

$$(\mathcal{F}_{(m)}, \mathcal{F}_{(n)}) = 0 \quad \text{if } m \neq n$$

For completeness we just mention (and encourage the reader to do the straightforward calculation) that for an $\mathfrak{su}(1, 1)$ -descendant state $\psi^{(N)} \equiv \frac{1}{N!} (L_{(-1)})^N \psi = \psi_{-N-1} \mathbf{1}$ the adjoint is given by

$$(\psi_n^{(N)})^\dagger = \sum_{i=0}^N (-1)^{\Delta_\psi+i} \binom{2\Delta_\psi+i}{N-i} \psi_{-n+N+i+2\Delta_\psi-2}^{(i)}$$

in agreement with (54) for $N = 0$.

To check whether adjointness satisfies the involution property $\mathcal{V}^{\dagger\dagger} = \mathcal{V}$ we note that the commutation relation $[L_{(0)}, L_{(n)}] = -n L_{(n)}$ yields $z_0^{L_{(0)}} L_{(n)} z_0^{-L_{(0)}} = z_0^{-n} L_{(n)}$ which by iteration gives us the conjugation formula

$$z_0^{L_{(0)}} e^{zL_{(n)}} z_0^{-L_{(0)}} = e^{z_0^{-n} z L_{(n)}}$$

so that we have indeed

$$\mathcal{V}\left(e^{z^{-1}L_{(1)}}(-z^2)^{L_{(0)}} e^{zL_{(1)}}(-z^{-2})^{L_{(0)}}\psi, z\right) = \mathcal{V}(\psi, z) \quad \forall \psi \in \mathcal{F}$$

It is by no means obvious from the definition that the bilinear form is symmetric. To establish symmetry we first note that $(\chi, \varphi) = \text{Res}_z [z^{-1}(\mathcal{V}(\chi, z)\mathbf{1}, \varphi)]$ by (41). Therefore it is sufficient to prove that $(\mathcal{V}(\chi, z)\mathbf{1}, \varphi) = (\varphi, \mathcal{V}(\chi, z)\mathbf{1})$. Now,

$$\begin{aligned}
(\mathcal{V}(\chi, z)\mathbf{1}, \varphi) &= (\mathbf{1}, \mathcal{V}(e^{zL_{(1)}}(-z^{-2})^{L_{(0)}}\chi, z^{-1})\varphi) \quad \text{by definition} \\
&= (\mathbf{1}, e^{z^{-1}L_{(-1)}}\mathcal{V}(\varphi, -z^{-1})e^{zL_{(1)}}(-z^{-2})^{L_{(0)}}\chi) \quad \text{by skew-symmetry} \\
&= (\mathcal{V}(e^{-z^{-1}L_{(1)}}(-z^2)^{L_{(0)}}\varphi, -z)\mathbf{1}, e^{zL_{(1)}}(-z^{-2})^{L_{(0)}}\chi) \quad \text{by involution} \\
&= (\mathcal{V}(\mathbf{1}, z)e^{-z^{-1}L_{(1)}}(-z^2)^{L_{(0)}}\varphi, (-z^{-2})^{L_{(0)}}\chi) \quad \text{by skew-symmetry} \\
&= (\varphi, (-z^2)^{L_{(0)}}e^{-z^{-1}L_{(-1)}}(-z^{-2})^{L_{(0)}}\chi) \quad \text{by definition} \\
&= (\varphi, \mathcal{V}(\chi, z)\mathbf{1}) \quad \text{by conjugation}
\end{aligned}$$

3.4 Algebras of primary fields of weight one

We shall provide a certain subspace of the Fock space \mathcal{F} with the structure of a Lie algebra. (cf. [11], [10], [32])

Looking at the skew-symmetry property (40) we can define a product by

$$[\psi, \varphi] := \psi_0\varphi \quad (55)$$

which is antisymmetric on the subspace $\mathcal{F}/L_{(-1)}\mathcal{F}$. Then, as already mentioned at the end of 3.1, the classical Jacobi identity for Lie algebras,

$$[[\psi, \varphi], \xi] + [[\varphi, \xi], \psi] + [[\xi, \psi], \varphi] = 0 \quad (56)$$

follows from the Jacobi identity for vertex algebras.

Another glimpse at skew-symmetry shows that the Lie algebra $\mathcal{F}/L_{(-1)}\mathcal{F}$ is also equipped with a symmetric product by

$$\langle \psi, \varphi \rangle := \psi_1\varphi \quad (57)$$

To investigate possible $\mathcal{F}/L_{(-1)}\mathcal{F}$ -invariance of $\langle _, _ \rangle$ we note that

$$\begin{aligned}
\langle [\psi, \varphi], \xi \rangle &\equiv (\psi_0\varphi)_1\xi \\
&= \psi_0(\varphi_1\xi) - \varphi_1(\psi_0\xi) \quad \text{by (27)} \\
&\equiv [\psi, \langle \varphi, \xi \rangle] - \langle \varphi, [\psi, \xi] \rangle
\end{aligned}$$

Hence the product $\langle _, _ \rangle$ is in general *not* $\mathcal{F}/L_{(-1)}\mathcal{F}$ -invariant unless we make further assumptions.

For that purpose let us restrict our attention to the piece of conformal weight one, i.e. to $\mathcal{F}_{(1)}$. Then the space

$$\mathcal{F}_{(1)} / (L_{(-1)}\mathcal{F} \cap \mathcal{F}_{(1)}) = \mathcal{F}_{(1)} / L_{(-1)}\mathcal{F}_{(0)}$$

is a subalgebra of the Lie algebra $\mathcal{F}/L_{(-1)}\mathcal{F}$ and, by (37),

$$\langle \varphi, \xi \rangle \in \mathcal{F}_{(0)} \quad \text{for all } \varphi, \xi \in \mathcal{F}_{(1)}$$

If we now assume that $\mathcal{F}_{(0)}$ is one-dimensional then $\langle \varphi, \xi \rangle$, $\varphi, \xi \in \mathcal{F}_{(1)}$, is a scalar multiple of the vacuum and $L_{(-1)}\mathcal{F}_{(0)} = 0$ by (32). Thus $[\psi, \langle \varphi, \xi \rangle]$ is proportional to $\psi_0\mathbf{1}$ which vanishes

because of (42) and we have indeed established invariance of the scalar product.

In some cases we can say even more. Let us look again at the commutator formula (29):

$$\begin{aligned}
[\psi_m, \varphi_n] &= \sum_{i \geq 0} \binom{m}{i} (\psi_i \varphi)_{m+n-i} \\
&= (\psi_0 \varphi)_{m+n} + m(\psi_1 \varphi)_{m+n-1} + \sum_{i \geq 2} \binom{m}{i} (\psi_i \varphi)_{m+n-i} \\
&= ([\psi, \varphi])_{m+n} + m \langle \psi, \varphi \rangle \mathbf{1}_{m+n-1} + \sum_{i \geq 2} \binom{m}{i} (\psi_i \varphi)_{m+n-i} \\
&= ([\psi, \varphi])_{m+n} + m \langle \psi, \varphi \rangle \delta_{m+n,0} \text{id}_{\mathcal{F}} + \sum_{i \geq 2} \binom{m}{i} (\psi_i \varphi)_{m+n-i} \quad \text{by (24)}
\end{aligned}$$

for $\psi, \varphi \in \mathcal{F}_{(1)}$. Since all states $\psi_i \varphi$, $i \geq 2$, which occur in the sum on the right-hand side have negative conformal weight we conclude:

Theorem 2 :

If the weight zero piece, $\mathcal{F}_{(0)}$, of a vertex algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ is one-dimensional and the spectrum of the operator $L_{(0)}$ is nonnegative then the weight one piece, $\mathcal{F}_{(1)}$, is a Lie algebra with antisymmetric product $[\psi, \varphi] := \psi_0 \varphi$ and $\mathcal{F}_{(1)}$ -invariant bilinear form $\langle \psi, \varphi \rangle := \psi_1 \varphi$. The space

$$\hat{\mathcal{F}}_{(1)} := \{\psi_n | \psi \in \mathcal{F}_{(1)}, n \in \mathbb{Z}\} \oplus \{\text{id}_{\mathcal{F}}\}$$

provides a representation of the affinization of $\mathcal{F}_{(1)}$ on \mathcal{F} by

$$[\psi_m, \varphi_n] = ([\psi, \varphi])_{m+n} + m \langle \psi, \varphi \rangle \delta_{m+n,0} \text{id}_{\mathcal{F}}$$

In particular, $\mathcal{F}_{(1)}$ may be identified with the Lie algebra of operators $\{\psi_0 | \psi \in \mathcal{F}_{(1)}\}$ on \mathcal{F} since in the adjoint representation, $\mathcal{F}_{(1)}$ acts on itself as operators ψ_0 .

It is quite interesting that in physical applications such as string theory, two-dimensional statistical systems and two-dimensional quantum field theories *physical* considerations lead to the same condition on the spectrum of $L_{(0)}$ as in Theorem 2. In such theories $L_{(0)}$ is identified with the Hamiltonian so that above condition immediately translates into the postulate of the positivity of the energy.(see [40], e.g.)

The Lie algebra $\mathcal{F}_{(1)}/L_{(-1)}\mathcal{F}_{(0)}$ is still too large for further investigations. Equation (46) tells us that if ψ is a primary state of weight one then the corresponding operator ψ_0 commutes with the Virasoro algebra, so it preserves all the physical subspaces $\mathcal{P}_{(n)}$. In other words, it maps physical states into physical states. Hence it is natural to look in detail at the Lie algebra

$$\mathfrak{g}_{\mathcal{F}} := \mathcal{P}_{(1)} / (L_{(-1)}\mathcal{F}_{(0)} \cap \mathcal{P}_{(1)})$$

Note that if $\psi = L_{(-1)}\varphi \in L_{(-1)}\mathcal{F}_{(0)}$ for some $\varphi \in \mathcal{F}_{(0)}$ then $\psi_0 = (\varphi_{-2}\mathbf{1})_0 = 0$ by (42), (28)(with $l = -2$, $n = 0$) and (24).

Again, if $\mathcal{F}_{(0)}$ is one-dimensional then $\mathcal{P}_{(1)}$ is a Lie algebra with antisymmetric product and invariant bilinear form defined as above.

On the other hand let us consider the case where $L_{(0)}$ has nonnegative spectrum. Then

$$L_{(-1)}\mathcal{F}_{(0)} \subset \mathcal{P}_{(1)}$$

because

$$L_{(n)}L_{(-1)}\psi \in \mathcal{F}_{(1-n)}, \quad L_{(1)}L_{(-1)}\psi = L_{(-1)}L_{(1)}\psi \in L_{(-1)}\mathcal{F}_{(-1)} \quad \text{for } \psi \in \mathcal{F}_{(0)}$$

and we find that $\mathcal{P}_{(1)}/L_{(-1)}\mathcal{F}_{(0)}$ is a Lie algebra.

Borcherds [11],[7],[10] was led to his definition of generalized Kac-Moody algebras precisely by Lie algebras of type $\mathcal{P}_{(1)}/(L_{(-1)}\mathcal{F}_{(0)} \cap \mathcal{P}_{(1)})$ for vertex algebras associated with even Lorentzian lattices.

When defining the Lie algebra $\mathcal{F}/L_{(-1)}\mathcal{F}$ we had to divide out the space $L_{(-1)}\mathcal{F}$ for mathematical reasons. Surprisingly, there is also a physical explanation for that procedure.(cf. [42]) Suppose that \mathcal{F} is equipped with an inner product $(-, -)$ such that the operator $L_{(-n)}$ is the adjoint of $L_{(n)}$ (cf. end of last subsection!). Then

$$(L_{(-1)}\varphi, \psi) = (\varphi, L_{(1)}\psi) = 0 \quad \text{for all } \varphi \in \mathcal{F}, \psi \in \mathcal{P}_{(n)}, n \in \mathbb{Z}$$

i.e. the space $L_{(-1)}\mathcal{F}$ is orthogonal to all physical states. In particular, $L_{(-1)}\mathcal{F}_{(0)} \cap \mathcal{P}_{(1)}$ consists of "null" physical states, physical states orthogonal to all physical states including themselves. Hence the Lie algebra $\mathfrak{g}_{\mathcal{F}}$ is obtained from $\mathcal{P}_{(1)}$ by dividing out null physical states.

It turns out that there are additional null physical states in $\mathcal{P}_{(1)}$ if and only if the central charge takes the critical value $c = 26$, namely the space $(L_{(-2)} + \frac{3}{2}L_{(-1)}^2)\mathcal{P}_{(-1)}$. Evidently, this space is annihilated by $L_{(n)}$ for $n \geq 3$. Furthermore

$$\begin{aligned} L_{(1)}(L_{(-2)} + \frac{3}{2}L_{(-1)}^2)\mathcal{P}_{(-1)} &= [L_{(1)}, L_{(-2)} + \frac{3}{2}L_{(-1)}^2]\mathcal{P}_{(-1)} \\ &= 3(L_{(-1)} + L_{(0)}L_{(-1)} + L_{(-1)}L_{(0)})\mathcal{P}_{(-1)} \\ &= 0 \\ L_{(2)}(L_{(-2)} + \frac{3}{2}L_{(-1)}^2)\mathcal{P}_{(-1)} &= [L_{(2)}, L_{(-2)} + \frac{3}{2}L_{(-1)}^2]\mathcal{P}_{(-1)} \\ &= (4L_{(0)} + \frac{c}{2} + \frac{9}{2}(L_{(0)}L_{(-1)} + L_{(-1)}L_{(0)}))\mathcal{P}_{(-1)} \\ &= 0 \quad \text{if and only if } c = 26 \end{aligned}$$

The existence of these additional null physical states is used in the proof of the No-ghost-theorem.[41],[42]

In applied conformal field theory one usually encounters the situation that the spectrum of $L_{(0)}$ is bounded below and the Fock space \mathcal{F} is equipped with a positive definite inner product $(-, -)$. Then a standard argument (see e.g. [36]) shows that in this case \mathcal{F} splits up into a direct sum of $\mathfrak{su}(1, 1)$ highest-weight representations generated by some basis of the quasi-primary states. Since for any quasi-primary state $\psi \in \mathcal{F}$ the $\mathfrak{su}(1, 1)$ representation space $[\psi] \subset \mathcal{F}$ generated from ψ is spanned by the elements of $\{(L_{(-1)})^N \psi | N \in \mathbb{N}\}$ we may identify $\mathcal{F}/L_{(-1)}\mathcal{F}$ with the set of quasi-primary states in the Fock space \mathcal{F} .

3.5 Cross-bracket and the algebra of fields of weight two

Before we leave the axiomatics of vertex algebras and turn to examples let us investigate the case where the vertex algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ contains *no* states of weight one.(see [32])

Define a product by

$$\psi \times \varphi := \psi_1 \varphi \quad \text{for } \psi, \varphi \in \mathcal{F}_{(2)}$$

which is symmetric but non-associative on the piece $\mathcal{F}_{(2)}$ in view of (40) with $L_{(-1)}\mathcal{F}_{(1)} = 0$. Note that $\frac{1}{2}\boldsymbol{\omega}$ provides an identity element on $\mathcal{F}_{(2)}$,

$$\frac{1}{2}\boldsymbol{\omega} \times \psi = \frac{1}{2}\boldsymbol{\omega}_1 \psi = \frac{1}{2}L_{(0)}\psi = \psi \quad \forall \psi \in \mathcal{F}_{(2)}$$

If we assume that $\mathcal{F}_{(0)}$ is one-dimensional then $\psi_3\varphi$ for $\psi, \varphi \in \mathcal{F}_{(2)}$ is a scalar multiple of the vacuum by (37) so that

$$\langle \psi, \varphi \rangle := \psi_3\varphi$$

gives us a symmetric bilinear form on $\mathcal{F}_{(2)}$. Moreover, this form is associative in the sense that

$$\langle \varphi, \psi \times \xi \rangle = \langle \varphi \times \psi, \xi \rangle \quad \text{for } \psi, \varphi, \xi \in \mathcal{F}_{(2)}$$

This can be seen most easily by setting $l = m = n = 1$ in (27):

$$\psi_2(\varphi_2\xi) - \varphi_2(\psi_2\xi) - \psi_1(\varphi_3\xi) + \varphi_3(\psi_1\xi) = (\psi_1\varphi)_3\xi + (\psi_2\varphi)_2\xi$$

Since $\mathcal{F}_{(1)} = 0$ the first two terms on the left-hand side and the last term on the right-hand side vanish while $\psi_1(\varphi_3\xi)$ is proportional to $\psi_1\mathbf{1}$ which is zero because of (42).

We define the **cross-bracket** as follows:

$$[\psi_m \times_1 \varphi_n] \equiv [\psi \times_1 \varphi]_{mn} := [\psi_{m+1}, \varphi_n] - [\psi_m, \varphi_{n+1}] \quad \text{for } \psi, \varphi \in \mathcal{F}_{(2)}$$

This looks a kind of awkward but turns out to be quite interesting as soon as one reminds the Jacobi identity in components, (27), which gives for $l = 1$, $\psi, \varphi \in \mathcal{F}_{(2)}$:

$$\begin{aligned} [\psi \times_1 \varphi]_{mn} &= \sum_{i \geq 0} \binom{m}{i} (\psi_{i+1}\varphi)_{m+n-i} \\ &= (\psi_1\varphi)_{m+n} + m \underbrace{(\psi_2\varphi)_{m+n-1}}_{=0} + \frac{1}{2}m(m-1)(\psi_3\varphi)_{m+n-2} + \sum_{i \geq 3} \binom{m}{i} (\psi_{i+1}\varphi)_{m+n-i} \\ &= (\psi \times \varphi)_{m+n} + \frac{1}{2}m(m-1) \langle \psi, \varphi \rangle \delta_{m+n,1} \text{id}_{\mathcal{F}} + \sum_{i \geq 3} \binom{m}{i} (\psi_{i+1}\varphi)_{m+n-i} \end{aligned}$$

Since the sum on the right-hand side involves only negative weight terms we have arrived at the following result:

Theorem 3 :

Let $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$ be a vertex algebra. If the weight zero piece is one-dimensional, the weight one piece is empty and the spectrum of the operator $L_{(0)}$ is nonnegative then the weight two piece, $\mathcal{F}_{(2)}$, is a non-associative algebra with symmetric product $\psi \times \varphi := \psi_1\varphi$ and associative bilinear form $\langle \psi, \varphi \rangle := \psi_3\varphi$. The space

$$\hat{\mathcal{F}}_{(2)} := \{ \psi_n \mid \psi \in \mathcal{F}_{(2)}, n \in \mathbb{Z} \} \oplus \{ \text{id}_{\mathcal{F}} \}$$

provides a representation of the commutative affinization of $\mathcal{F}_{(2)}$ on \mathcal{F} by the cross-bracket,

$$[\psi_m \times_1 \varphi_n] = (\psi \times \varphi)_{m+n} + \frac{1}{2}m(m-1) \langle \psi, \varphi \rangle \delta_{m+n,1} \text{id}_{\mathcal{F}}$$

Of course, this cross-bracket is quite a nice algebraic structure in our vertex algebra but immediately the question arises whether such a commutative non-associative algebra really exists. In fact, the so called Moonshine Module (see also Subsection 5.3) constructed by Frenkel, Lepowsky, and Meurman is a vertex operator algebra that satisfies all the assumptions of Theorem 3 and it turns out that then $\mathcal{F}_{(2)}$ is precisely the 196884-dimensional **Griess algebra** which possesses the Monster group, F_1 , as its full automorphism group.[43],[31],[30],[11]

3.6 Symmetry products

The commutator and the cross-bracket of the last two subsections can be embedded in an infinite family of symmetry products by using the Jacobi identity in the following way [32]: Take $\text{Res}_{z_0} [z_0^n (\text{Jacobi identity})]$ and define for $l \in \mathbb{Z}$

$$\begin{aligned} [\mathcal{V}(\psi, z_1) \times_l \mathcal{V}(\varphi, z_2)] &:= \text{Res}_{z_0} \left[z_0^l z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) \mathcal{V}(\mathcal{V}(\psi, z_0) \varphi, z_2) \right] \\ &= (z_1 - z_2)^l \mathcal{V}(\psi, z_1) \mathcal{V}(\varphi, z_2) - (-z_2 + z_1)^l \mathcal{V}(\varphi, z_2) \mathcal{V}(\psi, z_1) \end{aligned} \quad (58)$$

which is expressed in modes as

$$\begin{aligned} [\psi \times_l \varphi]_{mn} &:= \sum_{i \geq 0} (-1)^i \binom{l}{i} (\psi_{m+l-i} \varphi_{n+i} - (-1)^l \varphi_{n+l-i} \psi_{m+i}) \\ &= \sum_{i \geq 0} \binom{m}{i} (\psi_{l+i} \varphi)_{m+n-i} \end{aligned} \quad (59)$$

It is clear that these products are symmetric for odd l and alternating for even l . The symmetric product \times_1 is precisely the cross-bracket used to construct the affinization of the Griess algebra in Theorem 3 while the alternating product \times_0 yields nothing but the commutator for the modes in Theorem 2, $[\psi \times_0 \varphi]_{mn} = [\psi_m, \varphi_n]$. As the interpretation of the other symmetry products regards so far we can only associate with the product \times_{-1} a well-known feature of conformal field theory. Recall that the commutator of two fields is completely determined by the singular part of the operator product expansion. The regular part of the latter encodes the normal ordered product of two fields. Thus the product \times_{-1} which differs from \times_0 by a factor $(z_1 - z_2)^{-1}$ (more precisely, by a factor z_0^{-1} in $\text{Res}_{z_0} [\dots]$) might be a good guess for defining normal ordered products in a vertex algebra. For this purpose one would usually consider the (algebraic) limit $z_1 \rightarrow z_2$ of $[\mathcal{V}(\psi, z_1) \times_{-1} \mathcal{V}(\varphi, z_2)]$ which unfortunately does not exist. Hence we start with the following definition of **normal ordered product**:

$$\begin{aligned} \times \mathcal{V}(\psi, z) \mathcal{V}(\varphi, z) \times &:= \sum_{n \in \mathbb{Z}} [\psi \times_{-1} \varphi]_{0n} z^{-n-1} \\ &= \sum_{n \in \mathbb{Z}} (\psi_{-1} \varphi)_n z^{-n-1} \\ &= \mathcal{V}(\psi_{-1} \varphi, z) \end{aligned} \quad (60)$$

In the following we will also refer to the product $\psi_{-1} \varphi$ as normal ordered product of states. At first sight this does not look like the standard normal ordered product of fields in conformal field theory. However, using the mode expansion (59) we can rewrite the definition as

$$\begin{aligned} \times \mathcal{V}(\psi, z) \mathcal{V}(\varphi, z) \times &= \sum_{n \in \mathbb{Z}} \sum_{i \geq 0} (\psi_{-i-1} \varphi_{n+i} + \varphi_{n-i-1} \psi_i) z^{-n-1} \\ &= \sum_{n \in \mathbb{Z}} \sum_{i \geq 0} : \psi_{-i-1} \varphi_{n+i} : z^{-n-1} \end{aligned} \quad (61)$$

where we introduced the normal ordering of the modes,

$$: \psi_{-i-1} \varphi_{n+i} : := \begin{cases} \psi_{-i-1} \varphi_{n+i} & \text{if } i \geq 0 \\ \varphi_{n+i} \psi_{-i-1} & \text{if } i < 0 \end{cases}$$

And this is indeed the familiar normal ordered product of modes [37] if we employ the standard shifted grading $\psi_{(n)} \equiv \psi_{n+\Delta_\psi-1}$.

In general one expects the normal ordered product of bosonic fields to be commutative. Since skew-symmetry (40) forces $\psi_{-1}\varphi = \varphi_{-1}\psi$ on $\mathcal{F}/\mathcal{L}_{(-1)}\mathcal{F}$, we therefore should restrict above definition of normal ordered product to the quotient space $\mathcal{F}/\mathcal{L}_{(-1)}\mathcal{F}$. This has the nice effect of automatically projecting the normal ordered products of quasi-primary fields onto the space of quasi-primary fields. The idea is to subtract expressions of the form $(\mathcal{L}_{(-1)})^i(\psi_{i-1}\varphi)$, $i \geq 1$, from $\psi_{-1}\varphi$ such that one ends up with a quasi-primary state. Indeed, we found that the projected normal ordered product of two quasi-primary states ψ, φ is given by

$$[\psi *_{-1} \varphi] := \psi_{-1}\varphi + \sum_{i \geq 1} \frac{(-1)^i}{i!} \binom{2\Delta_\psi - 1}{i} \binom{2(\Delta_\psi + \Delta_\varphi - 1)}{i}^{-1} (\mathcal{L}_{(-1)})^i(\psi_{i-1}\varphi)$$

i.e. $[\psi *_{-1} \varphi]$ is a quasi-primary state of weight $\Delta_\psi + \Delta_\varphi$. This formula is similar to those given in [13] and [5]. For the sake of completeness we would like to mention that this projection onto quasi-primary states generalizes to any product $\psi_n\varphi$, $n \in \mathbb{Z}$, of two quasi-primary states, i.e.

$$[\psi *_l \varphi] := \sum_{i \geq 0} \frac{(-1)^i}{i!} p_i^{(l)}(\Delta_\psi, \Delta_\varphi) (\mathcal{L}_{(-1)})^i(\psi_{i+l}\varphi)$$

with

$$p_i^{(l)}(\Delta_\psi, \Delta_\varphi) := \binom{2\Delta_\psi - 2 - l}{i} \binom{2(\Delta_\psi + \Delta_\varphi - 2 - l)}{i}^{-1}$$

is a quasi-primary state of weight $\Delta_\psi + \Delta_\varphi - l - 1$. If we calculate the corresponding vertex operator using the translation axiom (20) we find

$$\mathcal{V}([\psi *_l \varphi], z) = \sum_{m \in \mathbb{Z}} \sum_{i \geq 0} p_i^{(l)}(\Delta_\psi, \Delta_\varphi) \binom{m}{i} (\psi_{l+i}\varphi)_{m-i}$$

Surprisingly, we observe that these projected products are just the “old” symmetry products (59) for $n = 0$ where each term in the sum is weighted with an additional polynomial factor $p_i^{(l)}(\Delta_\psi, \Delta_\varphi)$. Moreover we can prove that the projected products also inherit the symmetry properties from the original ones, i.e. $[\psi *_l \varphi] = (-1)^{l+1}[\varphi *_l \psi]$.

It is quite interesting that the normal ordered product (60) turns out to be also associative if we consider the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$. To see this we first note that $\mathcal{L}_{(-1)}\mathcal{F} = \mathcal{F}_{(-2)}\mathbf{1} \subset \mathcal{F}_{(-2)}\mathcal{F}$ by (42). Additionally we have for $n \leq -2$,

$$\begin{aligned} \psi_n\varphi &= \psi_n(\varphi_{-1}\mathbf{1}) = [\psi_n, \varphi_{-1}]\mathbf{1} + \varphi_{-1}(\psi_n\mathbf{1}) \\ &= \sum_{i \geq 0} \binom{n}{i} (\psi_i\varphi)_{n-1-i}\mathbf{1} + \frac{1}{(-n-1)!} \varphi_{-1}((\mathcal{L}_{(-1)})^{-n-1}\psi) \\ &= \underbrace{\sum_{i \geq 0} \frac{1}{(i-n)!} \binom{n}{i} ((\mathcal{L}_{(-1)})^{i-n-1}(\psi_i\varphi))_{-2}\mathbf{1}}_{\in \mathcal{F}_{(-2)}\mathbf{1}} + \underbrace{\frac{1}{(-n-1)!} \varphi_{-1}(((\mathcal{L}_{(-1)})^{-n-2}\psi)_{-2}\mathbf{1})}_{\in \mathcal{F}_{(-2)}\mathcal{F}} \end{aligned}$$

where the last term lies in $\mathcal{F}_{(-2)}\mathcal{F}$ because of

$$\varphi_{-1}(\xi_{-2}\mathbf{1}) = \xi_{-2}\varphi + \sum_{i \geq 0} \frac{(-1)^i}{(i+2)!} ((\mathcal{L}_{(-1)})^{i+1}(\varphi_i\xi))_{-2}\mathbf{1} \quad \forall \varphi, \xi \in \mathcal{F}$$

This gives us associativity of the normal ordered product,

$$\begin{aligned} (\psi_{-1}\varphi)_{-1}\chi - \psi_{-1}(\varphi_{-1}\chi) &= \sum_{i \geq 0} \{\psi_{-1-i}(\varphi_{-1+i}\chi) + \varphi_{-2-i}(\psi_i\chi)\} - \psi_{-1}(\varphi_{-1}\chi) \\ &= 0 \quad \text{on } \mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F} \end{aligned}$$

The fact that $\psi_n\varphi \in \mathcal{F}_{(-2)}\mathcal{F}$ for $n \leq -2$ in particular implies $L_{(n)}\mathcal{F} \subset \mathcal{F}_{(-2)}\mathcal{F}$ for $n \leq -3$ which allows us to give a nice interpretation of the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$. Consider a conformal family $[\psi]$ associated with a primary state $\psi \in \mathcal{F}$. It is spanned by elements of the form [4]

$$(L_{(-1)})^{i_1}(L_{(-2)})^{i_2} \dots (L_{(-n)})^{i_n}\psi \quad n \geq 1, i_1, \dots, i_n \geq 0$$

Hence only the subspace spanned by the states $(L_{(-2)})^N\psi$, $N \in \mathbb{N}$, survives when the conformal family is projected onto $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$. Moreover, $L_{(-2)} \equiv \omega_{-1}$ so that these states are just multiple normal ordered products of the Virasoro vector with the conformal highest weight vector ψ . We conclude: The quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ with the induced normal ordered product $\psi_{-1}\varphi$ carries the structure of a commutative associative algebra. If the Fock space \mathcal{F} splits up into a direct sum of highest weight representations of the Virasoro algebra then $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ can be identified with the span of primary states, Virasoro vector, and multiple normal ordered products of the latter with the primary states.

Above quotient space plays a special role when vertex algebras on the torus are considered i.e. when the notion of modular invariance is built into the framework of vertex algebras [74]. A vertex operator algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$ is said to satisfy the **finiteness condition** if the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ is finite-dimensional. A vertex operator algebra is called rational if it has only finitely many irreducible representations and every finitely generated representation is completely reducible. In fact, Zhu [74] proved that if a rational vertex operator algebra satisfies the finiteness condition then the linear span of the characters $\text{tr } q^{L_{(0)} - \frac{c}{24}}$ of its irreducible representations is modular invariant with respect to $\text{SL}(2, \mathbb{Z})$. It is conjectured that rational vertex operator algebras automatically satisfy the finiteness condition.

Finally we would like to mention that above discussion of the quotient space $\mathcal{F}/L_{(-1)}\mathcal{F}$ is especially useful for explaining how Gerstenhaber algebras arise in the context of super vertex algebras. To see this, one supposes that an odd operator Q satisfying $Q^2 = 0$, acts on the vertex algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$, and that Q can be represented as the zero mode of a vertex operator $\mathcal{V}(\sigma, z)$ associated with an odd “ghost” state σ of weight minus one, i.e. $Q = \sigma_0$. Furthermore one assumes that there is an odd “antighost” state β of weight two such that $L_{(n)} \equiv \omega_{n+1} = (Q\beta)_{n+1} \equiv (\sigma_0\beta)_{n+1} \forall n$. It is proved in [53] and [59] that the superspace $\ker Q/\text{im } Q$, the cohomology of Q , can be equipped with the structure of a Gerstenhaber algebra. It turns out that the dot product in this superspace is precisely given by $\psi_{-1}\varphi$. We observe that the structure of $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ is closely related to that of the cohomology of the nilpotent operator Q if we collect above formulas for the product $\psi_{-1}\varphi$ and add the properties of the bracket operation (55),(56),(30),(37) to obtain

Theorem 4 :

The dot product $\psi \cdot \varphi := \psi_{-1}\varphi$ and the bracket $[\psi, \varphi] := \psi_0\varphi$ enjoy the following properties on the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$,

- (i) $\psi \cdot \varphi = \varphi \cdot \psi$
- (ii) $(\psi \cdot \varphi) \cdot \xi = \psi \cdot (\varphi \cdot \xi)$

$$(iii) \quad [\psi, \varphi] = -[\varphi, \psi]$$

$$(iv) \quad [[\psi, \varphi], \xi] + [[\varphi, \xi], \psi] + [[\xi, \psi], \varphi] = 0$$

$$(v) \quad [\psi, \varphi \cdot \xi] = [\psi, \varphi] \cdot \xi + \varphi \cdot [\psi, \xi]$$

$$(vi) \quad [_, _] : \mathcal{F}_{(m)} / (\mathcal{F}_{(-2)} \mathcal{F} \cap \mathcal{F}_{(m)}) \times \mathcal{F}_{(n)} / (\mathcal{F}_{(-2)} \mathcal{F} \cap \mathcal{F}_{(n)}) \longrightarrow \mathcal{F}_{(m+n-1)} / (\mathcal{F}_{(-2)} \mathcal{F} \cap \mathcal{F}_{(m+n-1)})$$

Thus the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ may be thought of as part of some Gerstenhaber algebra [53]. In fact, the ghost number one part of the cohomology of Q corresponds precisely to the primary states of weight one which are also contained in $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$. We do not know yet how significant the resemblance of these two algebraic structures is.

4 Vertex algebras associated with even lattices

4.1 Statement of the theorem

It is by no means obvious that nontrivial examples of vertex (operator) algebras exist. However, a class of vertex algebras is provided by the following result.(see [11],[32],[22])

Theorem 5 :

Associated with each nondegenerate even lattice Λ there is a vertex algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$. If in addition Λ is positive definite then $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$ has the structure of a vertex operator algebra.

In fact, above examples of vertex algebras gave rise to the notion and the abstract definition of vertex algebras. Frenkel and Zhu [34] constructed vertex operator algebras corresponding to the highest weight representations of affine Lie algebras with the highest weights being multiples of the highest weight of the basic representation. Up to now these two classes are essentially all known examples of (untwisted) vertex operator algebras. Hence it is desirable to find other classes of examples of vertex algebras to which the general formalism may be applied. At present, however, research is concentrating on representation theory of vertex algebras [34],[22],[21],[20],[25],[26], generalizations of vertex algebras [23] and the geometric interpretation of vertex algebras [46],[45],[48].

The rest of this section we will be concerned with the explicit construction of the vertex algebra stated above.

4.2 Fock space and vertex operators

For physical motivations of the construction below the reader may consult the beautiful articles [39],[38],[40] or the comprehensive review [52].

Let Λ be an even lattice of rank $d < \infty$ with a symmetric nondegenerate \mathbb{Z} -valued \mathbb{Z} -bilinear form $_ \cdot _$ and corresponding metric tensor $\eta^{\mu\nu}$, $1 \leq \mu, \nu \leq d$ (Λ even means that $\mathbf{r}^2 \in 2\mathbb{Z}$ for all $\mathbf{r} \in \Lambda$). The vertex algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$ which we shall construct can be thought of as a bosonic string theory with d spacetime dimensions compactified on a torus. Thus Λ represents the allowed momentum vectors of the theory¹.

¹At this point we are rather sloppy since we should work with the complexified lattice $\Lambda_{\mathbb{C}} \equiv \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ rather than with the real lattice Λ itself. However, we believe that this subtlety is not essential for a physicist's understanding of the general construction.

Introduce orthonormal vectors ("zero mode states") $\Psi_{\mathbf{r}}, \mathbf{r} \in \Lambda$,

$$(\Psi_{\mathbf{r}}, \Psi_{\mathbf{s}}) = \delta_{\mathbf{rs}}$$

and oscillators $\alpha_m^\mu, m \in \mathbb{Z}, 1 \leq \mu \leq d$, satisfying the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0},$$

the hermiticity conditions

$$(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu,$$

and acting on zero mode states by

$$\begin{aligned} \alpha_m^\mu \Psi_{\mathbf{r}} &= 0 \quad \text{if } m > 0 \\ p^\mu \Psi_{\mathbf{r}} &= r^\mu \Psi_{\mathbf{r}} \end{aligned}$$

where $p^\mu \equiv \alpha_0^\mu$ and r^μ are the components of $\mathbf{r} \in \Lambda$. While the operators α_m^μ for $m > 0$ by definition act as annihilation operators, the creation operators $\alpha_m^\mu, m < 0$, generate the Fock space from the zero mode states. For convenience let us define

$$\mathbf{r}(m) := \sum_{\mu=1}^d r_\mu \alpha_m^\mu \equiv \mathbf{r} \cdot \boldsymbol{\alpha}_m$$

for $\mathbf{r} \in \Lambda, m \in \mathbb{Z}$, such that

$$[\mathbf{r}(m), \mathbf{s}(n)] = m(\mathbf{r} \cdot \mathbf{s})\delta_{m+n,0} \quad (62)$$

We denote the d -fold Heisenberg algebra spanned by the oscillators by

$$\hat{\mathbf{h}} := \{\mathbf{r}(m) | \mathbf{r} \in \Lambda, m \in \mathbb{Z}\}$$

and for the vector space of finite products of creation operators (\equiv algebra of polynomials on the oscillators) we write

$$S(\hat{\mathbf{h}}^-) := \bigoplus_{N \in \mathbb{N}} \left\{ \prod_{i=1}^N \mathbf{r}_i(-m_i) | \mathbf{r}_i \in \Lambda, m_i > 0 \text{ for } 1 \leq i \leq N \right\}$$

where "S" stands for "symmetric" because of the fact that the creation operators commute with each other.

If we introduce formally position operators $q^\mu, 1 \leq \mu \leq d$, commuting with α_m^μ for $m \neq 0$ and satisfying

$$[q^\nu, p^\mu] = i\eta^{\mu\nu}$$

then we find

$$e^{i\mathbf{r} \cdot \mathbf{q}} \Psi_{\mathbf{s}} = \Psi_{\mathbf{r+s}}$$

i.e. the zero mode states can be generated from the vacuum Ψ_0 :

$$\Psi_{\mathbf{r}} = e^{i\mathbf{r} \cdot \mathbf{q}} \Psi_0$$

Thus the operators $e^{i\mathbf{r} \cdot \mathbf{q}}, \mathbf{r} \in \Lambda$, may be identified with the zero mode states and form an abelian group which is called the group algebra of the lattice Λ and is denoted by $\mathbb{C}[\Lambda]$. One might expect the full Fock space \mathcal{F} of the vertex algebra to be $S(\hat{\mathbf{h}}^-) \otimes \mathbb{C}[\Lambda]$. However, it turns out

that we shall need to replace the group algebra $\mathbb{C}[\Lambda]$ by something more delicate in order to adjust some signs (in the Jacobi identity). We will multiply $e^{i\mathbf{r}\cdot\mathbf{q}}$ by a so called cocycle factor $c_{\mathbf{r}}$ which is a function of momentum \mathbf{p} . This means that it commutes with all oscillators α_m^μ and satisfies the eigenvalue equations

$$c_{\mathbf{r}}\Psi_{\mathbf{s}} = \epsilon(\mathbf{r}, \mathbf{s})\Psi_{\mathbf{s}}$$

Furthermore we define operators $e^{\mathbf{r}} := e^{i\mathbf{r}\cdot\mathbf{q}}c_{\mathbf{r}}$ and impose the conditions

$$e^{\mathbf{r}}e^{\mathbf{s}} = \epsilon(\mathbf{r}, \mathbf{s})e^{\mathbf{r}+\mathbf{s}} \quad (63)$$

$$e^{\mathbf{r}}e^{\mathbf{s}} = (-1)^{\mathbf{r}\cdot\mathbf{s}}e^{\mathbf{s}}e^{\mathbf{r}} \quad (64)$$

$$e^{\mathbf{r}}e^{-\mathbf{r}} = 1 \quad (65)$$

$$e^0 = 1 \quad (66)$$

which is equivalent to requiring, respectively,

$$\begin{aligned} \epsilon(\mathbf{r}, \mathbf{s})\epsilon(\mathbf{r} + \mathbf{s}, \mathbf{t}) &= \epsilon(\mathbf{r}, \mathbf{s} + \mathbf{t})\epsilon(\mathbf{s}, \mathbf{t}) \\ \epsilon(\mathbf{r}, \mathbf{s}) &= (-1)^{\mathbf{r}\cdot\mathbf{s}}\epsilon(\mathbf{s}, \mathbf{r}) \\ \epsilon(\mathbf{r}, -\mathbf{r}) &= 1 \\ \epsilon(\mathbf{0}, \mathbf{0}) &= 1 \end{aligned}$$

It is not difficult to show that it is always possible to construct cocycles with these properties. Note that every 2-cocycle $\epsilon : \Lambda \times \Lambda \rightarrow \{\pm 1\}$ corresponds to a central extension $\hat{\Lambda}$ of Λ by $\{\pm 1\}$:

$$1 \rightarrow \{\pm 1\} \rightarrow \hat{\Lambda} \rightarrow \Lambda \rightarrow 1$$

where we put $\hat{\Lambda} = \{\pm 1\} \times \Lambda$ as a set and define a multiplication in $\hat{\Lambda}$ by

$$(\rho, \mathbf{r}) * (\sigma, \mathbf{s}) := (\epsilon(\mathbf{r}, \mathbf{s})\rho\sigma, \mathbf{r} + \mathbf{s}) \quad \text{for } \rho, \sigma \in \{\pm 1\}, \mathbf{r}, \mathbf{s} \in \Lambda$$

We will take the twisted group algebra $\mathbb{C}\{\Lambda\}$ consisting of the operators $e^{\mathbf{r}}$, $\mathbf{r} \in \Lambda$, instead of $\mathbb{C}[\Lambda]$. This means nothing but working with the double cover $\hat{\Lambda}$ of the lattice Λ .

We summarize: The Fock space associated with the lattice Λ is defined to be

$$\mathcal{F} := S(\hat{\mathbf{h}}^-) \otimes \mathbb{C}\{\Lambda\}$$

Note that the oscillators $\mathbf{r}(m)$, $m \neq 0$, act only on the first tensor factor, namely, creation operators as multiplication operators and annihilation operators via the adjoint representation i.e. by (62). The zero mode operators α_0^μ , however, are only sensible for the twisted group algebra,

$$\mathbf{r}(0)e^{\mathbf{s}} \equiv (\mathbf{r} \cdot \mathbf{p})e^{\mathbf{s}} = (\mathbf{r} \cdot \mathbf{s})e^{\mathbf{s}} \quad \text{for all } \mathbf{r}, \mathbf{s} \in \Lambda \quad (67)$$

while the action of $e^{\mathbf{r}}$ on $\mathbb{C}\{\Lambda\}$ is given by (63).

We shall define next the (untwisted) vertex operators $\mathcal{V}(\psi, z)$ for $\psi \in \mathcal{F}$. For $\mathbf{r} \in \Lambda$ we introduce the formal sum

$$\mathbf{r}(z) := \sum_{m \in \mathbb{Z}} \mathbf{r}(m)z^{-m-1} \quad (68)$$

which is an element of $\hat{\mathbf{h}}[[z, z^{-1}]]$ and may be regarded as a generating function for the operators $\mathbf{r}(m)$, $m \in \mathbb{Z}$, or as a "currents" in contrast to the "states" in \mathcal{F} . It is convenient to split the current $\mathbf{r}(z)$ into three parts:

$$\mathbf{r}(z) = \mathbf{r}_<(z) + \mathbf{r}(0) + \mathbf{r}_>(z)$$

where

$$\mathbf{r}_<(z) := \sum_{m>0} \mathbf{r}(-m)z^{m-1}, \quad \mathbf{r}_>(z) := \sum_{m>0} \mathbf{r}(m)z^{-m-1}$$

We will employ the usual normal ordering procedure, i.e. colons indicate that in the enclosed expressions, q^ν is written to the left of p^μ , as well as the creation operators are to be placed to the left of the annihilation operators.

For $e^{\mathbf{r}} \in \mathbb{C}\{\Lambda\}$, we set

$$\mathcal{V}(e^{\mathbf{r}}, z) := e^{\int \mathbf{r}_<(z)dz} e^{\mathbf{r}} z^{(0)} e^{\int \mathbf{r}_>(z)dz} \quad (69)$$

using an obvious formal integration notation, i.e.

$$\begin{aligned} \int \mathbf{r}_<(z)dz &:= \sum_{m>0} \frac{1}{m} \mathbf{r}(-m)z^m \\ \int \mathbf{r}_>(z)dz &:= - \sum_{m>0} \frac{1}{m} \mathbf{r}(m)z^{-m} \end{aligned}$$

This can be written in a way more familiar to physicists by introducing formally the Fubini-Veneziano field,

$$Q^\mu(z) \equiv q^\mu - ip^\mu \ln z + i \sum_{m \in \mathbb{Z}} \frac{1}{m} \alpha_m^\mu z^{-m}$$

which really only has a meaning when exponentiated. We find

$$\mathcal{V}(e^{\mathbf{r}}, z) = :e^{i\mathbf{r} \cdot \mathbf{Q}(z)}: c_{\mathbf{r}}$$

Let $\psi = \left(\prod_{j=1}^N \mathbf{s}_j(-n_j) \right) \otimes e^{\mathbf{r}}$ be a typical homogeneous element of \mathcal{F} and define

$$\begin{aligned} \mathcal{V}(\psi, z) &:= : \mathcal{V}(e^{\mathbf{r}}, z) \prod_{j=1}^N \frac{1}{(n_j - 1)!} \left(\frac{d}{dz} \right)^{n_j-1} \mathbf{s}_j(z) : \\ &\equiv i : e^{i\mathbf{r} \cdot \mathbf{Q}(z)} \prod_{j=1}^N \frac{1}{(n_j - 1)!} \left(\frac{d}{dz} \right)^{n_j} (\mathbf{s}_j \cdot \mathbf{Q}(z)) : c_{\mathbf{r}} \end{aligned} \quad (70)$$

where we used $\frac{d}{dz}(i\mathbf{s} \cdot \mathbf{Q}(z)) = \mathbf{s}(z)$.

Extending this definition by linearity we finally obtain a well-defined map

$$\begin{aligned} \mathcal{V} : \mathcal{F} &\rightarrow (\text{End } \mathcal{F})[\![z, z^{-1}]\!] \\ \psi &\mapsto \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1} \end{aligned}$$

4.3 Regularity, vacuum, injectivity, conformal vector

We shall prove the first four axioms in the definition of a vertex algebra.

1. (Regularity)

Note that \mathcal{F} contains only states with a finite number of creation operators and the vertex operators are normal ordered expressions. Having this in mind it is clear that $\psi_n \varphi = 0$ for n large enough (depending on $\psi, \varphi \in \mathcal{F}$) since annihilation operators are always attached to negative powers of the formal variables.

2. (Vacuum)

We choose the vacuum to be the zero mode state with no momentum and without any creation operators, i.e.

$$\mathbf{1} := 1 \otimes e^0$$

so that

$$\mathcal{V}(\mathbf{1}, z) = :e^{i\mathbf{0} \cdot \mathbf{Q}(z)}: c_0 = \text{id}_{\mathcal{F}}$$

by the normalization condition (66).

3. (Injectivity)

Observe that, when acting on the vacuum, terms involving only creation operators survive in the expression for a vertex operator. Then it is obvious that

$$\psi_{-1}\mathbf{1} = \text{Res}_z \left[z^{-1} \mathcal{V}(\psi, z) \mathbf{1} \right] = \psi \quad \text{for all } \psi \in \mathcal{F}$$

In particular, $\mathcal{V}(\psi, z) = 0$ implies $\psi = 0$.

4. (Conformal vector)

We claim that the element

$$\boldsymbol{\omega} := \frac{1}{2} \sum_{\mu, \nu=1}^d \mathbf{e}_\mu(-1) \mathbf{e}_\nu(-1) \eta^{\mu\nu} (\otimes e^0)$$

is a conformal vector of dimension d and is independent of the choice of the basis $\{\mathbf{e}_\mu\}$ of Λ . We have

$$\begin{aligned} \mathcal{V}(\boldsymbol{\omega}, z) &= \frac{1}{2} \sum_{\mu, \nu=1}^d : \mathbf{e}_\mu(z) \mathbf{e}_\nu(z) : \eta^{\mu\nu} \quad \text{by (70)} \\ &= \frac{1}{2} \sum_{m, n \in \mathbb{Z}} : \boldsymbol{\alpha}_m \cdot \boldsymbol{\alpha}_n : z^{-m-n-2} \quad \text{by (68)} \end{aligned}$$

(Note that in the last step we had to rely on nondegeneracy of the lattice!) Thus

$$L_{(n)} \equiv \boldsymbol{\omega}_{n+1} = \frac{1}{2} \sum_{m \in \mathbb{Z}} : \boldsymbol{\alpha}_m \cdot \boldsymbol{\alpha}_{n-m} :$$

in agreement with the well-known expression from string theory. Using the oscillator commutation relations one indeed finds that the $L_{(n)}$'s obey (19). (see [42] for the calculation) To establish the translation property of $L_{(-1)}$ we quickly get

$$\begin{aligned} L_{(-1)} e^r &= \mathbf{r}(-1) e^r \quad \text{by (67)} \\ L_{(-1)} \mathbf{r}(-m) &= m \mathbf{r}(-m-1) \quad \text{for } m > 0 \quad \text{by (62)} \end{aligned}$$

but on the other hand,

$$\begin{aligned} \frac{d}{dz} \mathcal{V}(e^r, z) &= : \mathbf{r}(z) \mathcal{V}(e^r, z) : = \mathcal{V}(\mathbf{r}(-1) e^r, z) \quad \text{by (70), (69)} \\ \frac{d}{dz} \mathcal{V}(\mathbf{r}(-m), z) &= \frac{1}{(m-1)!} \left(\frac{d}{dz} \right)^m \mathbf{r}(z) = \mathcal{V}(m \mathbf{r}(-m-1), z) \quad \text{by (70), (68)} \end{aligned}$$

Together with derivation property of $L_{(-1)}$ and $\frac{d}{dz}$ this proves (20).
Finally, consider a typical homogeneous element of \mathcal{F} ,

$$\psi = \left(\prod_{j=1}^N \mathbf{s}_j(-n_j) \right) \otimes e^{\mathbf{r}}$$

Then

$$\begin{aligned} L_{(0)}\psi &= \left\{ \frac{1}{2}\mathbf{p}^2 + \sum_{m \geq 1} \boldsymbol{\alpha}_{-m} \cdot \boldsymbol{\alpha}_m \right\} \left(\left(\prod_{j=1}^N \mathbf{s}_j(-n_j) \right) \otimes e^{\mathbf{r}} \right) \\ &= \left(\frac{1}{2}\mathbf{r}^2 + \sum_{j=1}^N n_j \right) \psi \end{aligned}$$

yields the desired grading of \mathcal{F} . Furthermore we observe that the spectrum of $L_{(0)}$ is nonnegative and the eigenspaces of $L_{(0)}$ are finite-dimensional provided that Λ is a positive definite lattice.

4.4 Jacobi identity

It is not surprising that by far the hardest axiom to prove is the Jacobi identity because it contains most information about a vertex algebra. We will not go much into details and will only mention the important steps and crucial ideas.(see [32])

Step 1:

We make a change of variables. Let $\mathbf{r}_1, \dots, \mathbf{r}_M, \mathbf{s}_1, \dots, \mathbf{s}_N \in \Lambda$ and consider the formal sums

$$\begin{aligned} R &\equiv \prod_{i=1}^M \left(e^{\sum_{m>0} \frac{1}{m} \mathbf{r}_i(-m) x_i^m} e^{\mathbf{r}_i} \right) \\ &= \prod_{i=1}^M \left(\sum_{m \geq 0} p_m(\mathbf{r}_i(-1), \dots, \mathbf{r}_i(-m)) x_i^m e^{\mathbf{r}_i} \right) \in \mathcal{F}[[x_1, \dots, x_M]] \\ S &\equiv \prod_{j=1}^N \left(e^{\sum_{n>0} \frac{1}{n} \mathbf{s}_j(-n) y_j^n} e^{\mathbf{s}_j} \right) \\ &= \prod_{j=1}^N \left(\sum_{n \geq 0} p_n(\mathbf{s}_j(-1), \dots, \mathbf{s}_j(-n)) y_j^n e^{\mathbf{s}_j} \right) \in \mathcal{F}[[y_1, \dots, y_N]] \end{aligned}$$

where the Schur polynomials $p_n(w_1, \dots, w_n)$ are

$$p_0 = 1, \quad p_1 = w_1, \quad p_2 = \frac{1}{2!}(w_1^2 + w_2), \quad p_3 = \frac{1}{3!}(w_1^3 + 3w_1w_2 + 2w_3), \quad \dots$$

We note that the coefficients of the monomials in the formal variables span \mathcal{F} as M and the \mathbf{r}_i 's and N and the \mathbf{s}_j 's vary, respectively. Hence it suffices to prove the Jacobi identity with ψ and φ replaced by R and S , respectively.

Using (69) and (67) we can immediately rewrite R and S as

$$\begin{aligned} R &= : \prod_{i=1}^M \mathcal{V}(e^{\mathbf{r}_i}, x_i) : \mathbf{1} \\ S &= : \prod_{j=1}^N \mathcal{V}(e^{\mathbf{s}_j}, y_j) : \mathbf{1} \end{aligned}$$

Step 2:

A lengthy but straightforward calculation which uses normal ordering properties and (12) shows that

$$\begin{aligned}\mathcal{V}(R, z_1) &= :e^{\sum_{i=1}^M \sum_{m \geq 1} \frac{1}{m!} \left(\frac{d}{dz_1}\right)^{m-1} \mathbf{r}_i(z_1) x_i^m} \mathcal{V}\left(\prod_{i=1}^M e^{\mathbf{r}_i}, z_1\right) := : \prod_{i=1}^M \mathcal{V}(e^{\mathbf{r}_i}, z_1 + x_i) : \\ \mathcal{V}(S, z_2) &= :e^{\sum_{j=1}^N \sum_{n \geq 1} \frac{1}{n!} \left(\frac{d}{dz_2}\right)^{n-1} \mathbf{s}_j(z_2) y_j^n} \mathcal{V}\left(\prod_{j=1}^N e^{\mathbf{s}_j}, z_2\right) := : \prod_{j=1}^N \mathcal{V}(e^{\mathbf{s}_j}, z_2 + y_j) : \end{aligned}$$

Hence

$$:\mathcal{V}(R, z_1)\mathcal{V}(S, z_2): = \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} (-1)^{\mathbf{r}_i \cdot \mathbf{s}_j} : \mathcal{V}(S, z_2)\mathcal{V}(R, z_1) :$$

and

$$\begin{aligned}\mathcal{V}(R, z_1)\mathcal{V}(S, z_2) &= \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} \left(z_1 + (-z_2 + x_i - y_j)\right)^{\mathbf{r}_i \cdot \mathbf{s}_j} : \mathcal{V}(R, z_1)\mathcal{V}(S, z_2) : \\ \mathcal{V}(S, z_2)\mathcal{V}(R, z_1) &= \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} \left(-z_2 + (z_1 + x_i - y_j)\right)^{\mathbf{r}_i \cdot \mathbf{s}_j} : \mathcal{V}(R, z_1)\mathcal{V}(S, z_2) : \end{aligned}$$

(all binomial expressions to be expanded in the second term!)

Step 3:

Fix $k \in \mathbb{Z}$ and a monomial $q = \prod_{i=1}^M \prod_{j=1}^N x_i^{m_i} y_j^{n_j}$, $m_i, n_j \geq 0 \ \forall i, j$. Choose $K \geq 0$ such that $K+k \geq 0$ and $K+k \geq \deg q - \sum_{i=1}^M \sum_{j=1}^N \mathbf{r}_i \cdot \mathbf{s}_j$. Thus the coefficient of q and of each monomial of lower total degree than q in

$$\begin{aligned}F_K &\equiv (z_1 - z_2)^{K+k} \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} \left(z_1 + (-z_2 + x_i - y_j)\right)^{\mathbf{r}_i \cdot \mathbf{s}_j} \\ &= (z_1 - z_2)^{K+k} \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} \sum_{\substack{m_i \geq 0 \\ n_j \geq 0}} \binom{\mathbf{r}_i \cdot \mathbf{s}_j}{m_i} \binom{\mathbf{r}_i \cdot \mathbf{s}_j - m_i}{n_j} (-1)^{n_j} (z_1 - z_2)^{\mathbf{r}_i \cdot \mathbf{s}_j - m_i - n_j} x_i^{m_i} y_j^{n_j} \end{aligned}$$

is a *polynomial* in $z_1 - z_2$.

Let $V_q(z_1, z_2)$ denote the coefficient of q in

$$(z_1 - z_2)^{K+k} \mathcal{V}(R, z_1)\mathcal{V}(S, z_2) = F_K : \mathcal{V}(R, z_1)\mathcal{V}(S, z_2) :$$

Then the coefficient of q in $(z_1 - z_2)^k \mathcal{V}(R, z_1)\mathcal{V}(S, z_2)$ is $(z_1 - z_2)^{-K} V_q(z_1, z_2)$. Similarly we find that the coefficient of q in $(-z_2 + z_1)^k \mathcal{V}(S, z_2)\mathcal{V}(R, z_1)$ is $(-z_2 + z_1)^{-K} V_q(z_1, z_2)$.

It follows that the coefficient of q in

$$(z_1 - z_2)^k \mathcal{V}(R, z_1)\mathcal{V}(S, z_2) - (-z_2 + z_1)^k \mathcal{V}(S, z_2)\mathcal{V}(R, z_1)$$

is, by (8),

$$-(-z_2)^{-K} \Theta\left(\left(1 - \frac{z_1}{z_2}\right)^{-K}\right) V_q(z_1, z_2)$$

which is the coefficient of z_0^{K-1} in

$$z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) V_q(z_1, z_2) = z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) V_q(z_2 + z_0, z_2)$$

by (8),(6) and (5).

But $V_q(z_2 + z_0, z_2)$ is the coefficient of q in

$$:\mathcal{V}(R, z_2 + z_0)\mathcal{V}(S, z_2): z_0^{K+k} \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} (z_0 + (x_i - y_j))^{\mathbf{r}_i \cdot \mathbf{s}_j}$$

Hence $(z_1 - z_2)^k \mathcal{V}(R, z_1)\mathcal{V}(S, z_2) - (-z_2 + z_1)^k \mathcal{V}(S, z_2)\mathcal{V}(R, z_1)$ is the coefficient of z_0^{-k-1} in

$$z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) :\mathcal{V}(R, z_2 + z_0)\mathcal{V}(S, z_2): \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} (z_0 + (x_i - y_j))^{\mathbf{r}_i \cdot \mathbf{s}_j}$$

Note that the last expression is independent of q and $K!$ We conclude that

$$\begin{aligned} z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) \mathcal{V}(R, z_1)\mathcal{V}(S, z_2) - z_0^{-1} \delta \left(\frac{-z_2 + z_1}{z_0} \right) \mathcal{V}(S, z_2)\mathcal{V}(R, z_1) \\ = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) :\mathcal{V}(R, z_2 + z_0)\mathcal{V}(S, z_2): \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} (z_0 + (x_i - y_j))^{\mathbf{r}_i \cdot \mathbf{s}_j} \end{aligned}$$

Step 4:

On the other hand, using *Step 2*,

$$\mathcal{V}(R, z_0)S = \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} (z_0 + (x_i - y_j))^{\mathbf{r}_i \cdot \mathbf{s}_j} : \prod_{i=1}^M \mathcal{V}(\mathbf{e}^{\mathbf{r}_i}, z_0 + x_i) \prod_{j=1}^N \mathcal{V}(\mathbf{e}^{\mathbf{s}_j}, y_j) : \mathbf{1}$$

Thus

$$\begin{aligned} \mathcal{V}(\mathcal{V}(R, z_0)S, z_2) &= \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} (z_0 + (x_i - y_j))^{\mathbf{r}_i \cdot \mathbf{s}_j} : \prod_{i=1}^M \mathcal{V}(\mathbf{e}^{\mathbf{r}_i}, z_2 + z_0 + x_i) \prod_{j=1}^N \mathcal{V}(\mathbf{e}^{\mathbf{s}_j}, z_2 + y_j) : \\ &= :\mathcal{V}(R, z_2 + z_0)\mathcal{V}(S, z_2): \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} (z_0 + (x_i - y_j))^{\mathbf{r}_i \cdot \mathbf{s}_j} \end{aligned}$$

This completes the proof of the Jacobi identity.

4.5 Example 1: Affine Lie algebras

Suppose that Λ is a positive definite even lattice. Obviously, $\mathcal{F}_{(0)}$ is one-dimensional and the spectrum of $\mathbf{L}_{(0)}$ is nonnegative so that $\mathcal{F}_{(1)}$ is a Lie algebra. Its elements are easy to describe,

$$\mathcal{F}_{(1)} = \{\mathbf{e}^{\mathbf{r}} | \mathbf{r} \in \Lambda_2\} \oplus \{\mathbf{s}(-1) | \mathbf{s} \in \Lambda\}$$

where $\Lambda_2 \equiv \{\mathbf{r} \in \Lambda | \mathbf{r}^2 = 2\}$ denotes the set of lattice vectors of squared length two. We can work out the antisymmetric product and the invariant bilinear form explicitly:

$$\begin{aligned} [\mathbf{r}(-1), \mathbf{s}(-1)] &= \mathbf{r}(-1)_0 \mathbf{s}(-1) = \text{Res}_z [\mathbf{r}(z)(\mathbf{s}(-1))] \\ &= \text{Res}_z \left[\sum_{m \in \mathbb{Z}} z^{-m-1} \mathbf{r}(m)(\mathbf{s}(-1)) \right] \\ &= 0 \quad \text{by (62)} \end{aligned}$$

$$\begin{aligned}
[\mathbf{r}(-1), \mathbf{e}^{\mathbf{s}}] &= \mathbf{r}(-1)_0 \mathbf{e}^{\mathbf{s}} = \text{Res}_z [\mathbf{r}(z)(\mathbf{e}^{\mathbf{s}})] \\
&= \text{Res}_z \left[\sum_{m \in \mathbb{Z}} z^{-m-1} \mathbf{r}(m)(\mathbf{e}^{\mathbf{s}}) \right] \\
&= (\mathbf{r} \cdot \mathbf{s}) \mathbf{e}^{\mathbf{s}} \quad \text{by (67)}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{e}^{\mathbf{r}}, \mathbf{e}^{\mathbf{s}}] &= \mathbf{e}_0^{\mathbf{r}} \mathbf{e}^{\mathbf{s}} = \text{Res}_z \left[e^{\int \mathbf{r}_{<}(z) dz} \mathbf{e}^{\mathbf{r}} z^{\mathbf{r}(0)} e^{\int \mathbf{r}_{>}(z) dz} (\mathbf{e}^{\mathbf{s}}) \right] \\
&= \text{Res}_z \left[e^{\sum_{m>0} \frac{1}{m} \mathbf{r}(-m) z^m} z^{\mathbf{r} \cdot \mathbf{s}} \mathbf{e}^{\mathbf{r}} \mathbf{e}^{\mathbf{s}} \right] \\
&= \begin{cases} 0 & \text{if } \mathbf{r} \cdot \mathbf{s} \geq 0 \\ \epsilon(\mathbf{r}, \mathbf{s}) \mathbf{e}^{\mathbf{r}+\mathbf{s}} & \text{if } \mathbf{r} \cdot \mathbf{s} = -1 \\ \mathbf{r}(-1) & \text{if } \mathbf{r} \cdot \mathbf{s} = -2 \end{cases}
\end{aligned}$$

We note that the Schwarz inequality yields $|\mathbf{r} \cdot \mathbf{s}| \leq 2$. Moreover $\mathbf{r} \cdot \mathbf{s} = -1 \iff \mathbf{r} + \mathbf{s} \in \Lambda_2$ and $\mathbf{r} \cdot \mathbf{s} = -2 \iff \mathbf{r} + \mathbf{s} = 0$ for $\mathbf{r}, \mathbf{s} \in \Lambda_2$.

Similarly, we find

$$\begin{aligned}
\langle \mathbf{r}(-1), \mathbf{s}(-1) \rangle &= \mathbf{r}(-1)_1 \mathbf{s}(-1) = \mathbf{r} \cdot \mathbf{s} \\
\langle \mathbf{r}(-1), \mathbf{e}^{\mathbf{s}} \rangle &= \mathbf{r}(-1)_1 \mathbf{e}^{\mathbf{s}} = 0 \\
\langle \mathbf{e}^{\mathbf{r}}, \mathbf{e}^{\mathbf{s}} \rangle &= \mathbf{e}_1^{\mathbf{r}} \mathbf{e}^{\mathbf{s}} = \begin{cases} 0 & \text{if } \mathbf{r} \cdot \mathbf{s} \geq -1 \\ 1 & \text{if } \mathbf{r} \cdot \mathbf{s} = -2 \end{cases}
\end{aligned}$$

Thus we have arrived at a root space decomposition of the Lie algebra $\mathcal{F}_{(1)}$ where the root lattice is precisely the lattice Λ and the set of roots is given by Λ_2 .

For the affine Lie algebra $\hat{\mathcal{F}}_{(1)}$ we find the formulas

$$\begin{aligned}
[\mathbf{r}(-1)_m, \mathbf{s}(-1)_n] &= m(\mathbf{r} \cdot \mathbf{s}) \delta_{m+n,0} \\
[\mathbf{r}(-1)_m, \mathbf{e}_n^{\mathbf{s}}] &= (\mathbf{r} \cdot \mathbf{s}) \mathbf{e}_{m+n}^{\mathbf{s}} \\
[\mathbf{e}_m^{\mathbf{r}}, \mathbf{e}_n^{\mathbf{s}}] &= \begin{cases} 0 & \text{if } \mathbf{r} \cdot \mathbf{s} \geq 0 \\ \epsilon(\mathbf{r}, \mathbf{s}) \mathbf{e}_{m+n}^{\mathbf{r}+\mathbf{s}} & \text{if } \mathbf{r} \cdot \mathbf{s} = -1 \\ \mathbf{r}(-1)_{m+n} + m \delta_{m+n,0} & \text{if } \mathbf{r} \cdot \mathbf{s} = -2 \end{cases}
\end{aligned}$$

The first equation is no surprise since the operators $\mathbf{r}(-1)_m$ are nothing but the oscillators we have started with,

$$\mathbf{r}(-1)_m = \text{Res}_z [z^m \mathbf{r}(z)] = \mathbf{r}(m)$$

while the operators $\mathbf{e}_m^{\mathbf{r}}$ are those occurring in the Frenkel-Kac construction [29] of affine Lie algebras,

$$\mathbf{e}_m^{\mathbf{r}} = \text{Res}_z [z^m : e^{i\mathbf{r} \cdot \mathbf{Q}(z)} :] c_{\mathbf{r}}$$

In physics literature this construction of affine Lie algebras is presented pedagogically in [65],[51],[39],[40],[38].

4.6 Example 2: Fake Monster Lie algebra

Things become more complicated when we move away from the lattice Λ being Euclidian. Let us consider the unique 26-dimensional even unimodular Lorentzian lattice $\mathbb{II}_{25,1}$, which can be taken to be the lattice of all points $\mathbf{x} = (x_1, \dots, x_{25} | x_0)$ for which the x_μ are all in \mathbb{Z} or all in $\mathbb{Z} + \frac{1}{2}$ and which have integer inner product with the vector $\mathbf{l} = (\frac{1}{2}, \dots, \frac{1}{2} | \frac{1}{2})$, all norms and inner products being evaluated in the Lorentzian metric $\mathbf{x}^2 = x_1^2 + \dots + x_{25}^2 - x_0^2$. (cf.

[64]) In physics this corresponds to an open bosonic string moving in 26-dimensional spacetime compactified on a torus so that the momenta lie on a lattice. [52] Calculations in connection with the automorphism group of $\mathbb{II}_{25,1}$ show that a set of positive norm simple roots for $\mathbb{II}_{25,1}$ is given by the subset of vectors \mathbf{r} in $\mathbb{II}_{25,1}$ for which $\mathbf{r}^2 = 2$ and $\mathbf{r} \cdot \boldsymbol{\rho} = -1$ where the Weyl vector is $\boldsymbol{\rho} = (0, 1, 2, \dots, 24|70)$. (cf. [16], [14]) In fact, these simple roots generate the reflection group of $\mathbb{II}_{25,1}$, where the reflection $\sigma_{\mathbf{r}}$ associated with a root \mathbf{r} is defined as $\sigma_{\mathbf{r}}(\mathbf{x}) = \mathbf{x} - \frac{2}{\mathbf{r}^2}(\mathbf{x} \cdot \mathbf{r})\mathbf{r}$. We shall also call the positive norm simple roots of $\mathbb{II}_{25,1}$ Leech roots since Conway has shown that this subset is indeed isometric to the Leech lattice, the unique 24-dimensional even unimodular Euclidian lattice with no vectors of square length two. For further informations about the Leech lattice and the other Niemeier lattices the reader may wish to consult [58], [15], [17], [8].

We now define a Kac-Moody algebra L_{∞} , of infinite dimension and rank, as follows (see [12]): L_{∞} has three generators $e(\mathbf{r})$, $f(\mathbf{r})$, $h(\mathbf{r})$ for each Leech root \mathbf{r} , and is presented by the following relations,

$$\begin{aligned} [h(\mathbf{r}), e(\mathbf{s})] &= (\mathbf{r} \cdot \mathbf{s})e(\mathbf{s}) \\ [h(\mathbf{r}), f(\mathbf{s})] &= -(\mathbf{r} \cdot \mathbf{s})f(\mathbf{s}) \\ [e(\mathbf{r}), f(\mathbf{r})] &= h(\mathbf{r}) \\ [e(\mathbf{r}), f(\mathbf{s})] &= 0 \\ [h(\mathbf{r}), h(\mathbf{s})] &= 0 \\ (\text{ad } e(\mathbf{s}))^{1-\mathbf{r} \cdot \mathbf{s}} e(\mathbf{r}) &= (\text{ad } f(\mathbf{s}))^{1-\mathbf{r} \cdot \mathbf{s}} f(\mathbf{r}) = 0 \end{aligned}$$

where \mathbf{r} and \mathbf{s} are distinct Leech roots. In a sort of Dynkin diagram for L_{∞} two nodes \mathbf{r}, \mathbf{s} are joined by $-\mathbf{r} \cdot \mathbf{s}$ lines and a portion of the (infinite) graph looks like the following figure (cf. [14]):

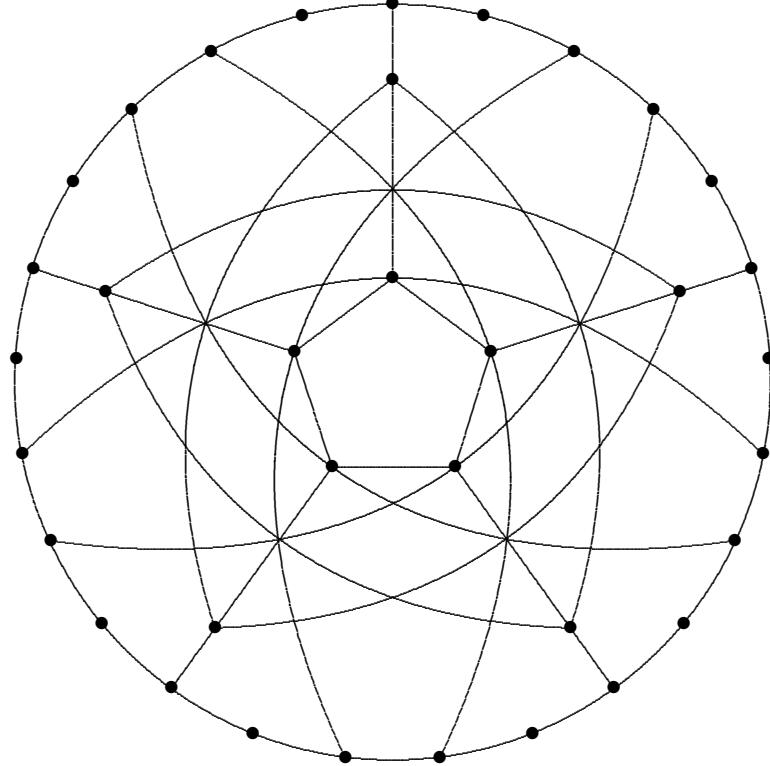


Figure 1: Part of the Dynkin diagram of L_{∞}

Besides infiniteness another fascinating feature of this graph is the obvious fivefold symmetry. We have checked the next (w.r.t. increasing value of the time coordinate) fifteen Leech roots

and found that they also fit nicely into this pentagram-like symmetry. If this pattern turned out to be true for the whole diagram then it should be somehow reflected in the structure of the automorphism group for the Leech lattice.

Frenkel [27] proved that the dimension of the root space corresponding to any root \mathbf{r} of L_∞ is *at most* $p_{24}(1 - \frac{1}{2}\mathbf{r}^2)$ where $p_{24}(n)$ is the number of partitions of $n \in \mathbb{N}$ into parts of 24 colours, i.e.

$$\Delta(q)^{-1} \equiv \sum_{n \geq 0} p_{24}(1 + n)q^n = q^{-1} \prod_{n > 0} (1 - q^n)^{-24} = q^{-1} + 24 + 324q + 3200q^2 + \dots$$

Let us return to the vertex algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ associated with the lattice $II_{25,1}$. It is easy to see that, for any Leech root \mathbf{r} , the elements $e^\mathbf{r}$, $e^{-\mathbf{r}}$, $\mathbf{r}(-1)$ describe the physical states of conformal weight one, i.e. they lie in $\mathcal{P}_{(1)}$. We can immediately infer from Example 1 that L_∞ is mapped into the Lie algebra $\mathfrak{g}_{II_{25,1}} := \mathcal{P}_{(1)}/\text{kernel}(-, -)$ by

$$\begin{aligned} e(\mathbf{r}) &\mapsto e^\mathbf{r} \\ f(\mathbf{r}) &\mapsto e^{-\mathbf{r}} \\ h(\mathbf{r}) &\mapsto \mathbf{r}(-1) \end{aligned}$$

Hence L_∞ is a subalgebra of $\mathfrak{g}_{II_{25,1}}$. Note that we have to divide out the kernel of $(-, -)$ since in 26 dimensions additional null physical states besides $L_{(-1)}\mathcal{F}_{(0)} \cap \mathcal{P}_{(1)}$ occur.

However this is not the whole story about $\mathfrak{g}_{II_{25,1}}$ since L_∞ is a *proper* subalgebra of $\mathfrak{g}_{II_{25,1}}$. To see this we exploit the connection with string theory as presented in [42]. So far we have only considered the tachyonic ground states $e^\mathbf{r}$ and the oscillators $\mathbf{r}(-1)_m \equiv \mathbf{r}(m)$. Hence we still have to add the DDF operators which correspond to photon emission vertices. In fact, the elements $\boldsymbol{\xi}(-1)e^{m\boldsymbol{\rho}}$ for $m \in \mathbb{Z}$, $\boldsymbol{\xi} \in II_{25,1}$, are physical states provided $\boldsymbol{\xi} \cdot \boldsymbol{\rho} = 0$, i.e. provided that $\boldsymbol{\xi}$ is a transverse polarization vector. $\boldsymbol{\xi}$ must not be proportional to the (lightlike!) Weyl vector $\boldsymbol{\rho}$ because otherwise the corresponding state would be a null physical state (more precisely, it would lie in $L_{(-1)}\mathcal{F}_{(1)} \cap \mathcal{P}_{(1)}$). This leaves us with 24 degrees of freedom for the polarization. Additionally we have to incorporate the states $m\boldsymbol{\rho}(-1)$ for $m \in \mathbb{Z}$. The calculation of the Lie bracket gives

$$\begin{aligned} [\mathbf{r}(-1), \boldsymbol{\xi}(-1)e^{m\boldsymbol{\rho}}] &= m(\mathbf{r} \cdot \boldsymbol{\rho})\boldsymbol{\xi}(-1)e^{m\boldsymbol{\rho}} \quad \forall \mathbf{r} \in II_{25,1} \\ [e^\mathbf{r}, \boldsymbol{\xi}(-1)e^{m\boldsymbol{\rho}}] &= 0 \quad \text{if } m(\mathbf{r} \cdot \boldsymbol{\rho}) \geq 1 \\ [\boldsymbol{\xi}(-1)e^{m\boldsymbol{\rho}}, \boldsymbol{\eta}(-1)e^{n\boldsymbol{\rho}}] &= m(\boldsymbol{\xi} \cdot \boldsymbol{\eta})\boldsymbol{\rho}(-1)\delta_{m+n,0} \end{aligned}$$

where we used $\boldsymbol{\xi} \cdot \boldsymbol{\rho} = \boldsymbol{\eta} \cdot \boldsymbol{\rho} = \boldsymbol{\rho} \cdot \boldsymbol{\rho} = 0$ and $\boldsymbol{\rho}(-1)e^{m\boldsymbol{\rho}}e^{n\boldsymbol{\rho}} = \frac{1}{m+n}L_{(-1)}(e^{m\boldsymbol{\rho}}e^{n\boldsymbol{\rho}}) \in L_{(-1)}\mathcal{F}_{(0)}$ if $m + n \neq 0$.

The no-ghost theorem together with Borcherds' theorem [9] tell us that this is a complete set of generators for the spectrum of physical states and that the dimension of a subspace of momentum $\mathbf{x} \in II_{25,1}$ is *exactly* given by $p_{24}(1 - \frac{1}{2}\mathbf{x}^2)$. Moreover, the bilinear form $(-, -)$ is positive definite on any subspace of nonzero momentum \mathbf{x} .

Let us summarize: We define the **fake Monster Lie algebra** $\mathfrak{g}_{II_{25,1}}$ to be the Lie algebra with root lattice $II_{25,1}$, whose simple roots are the simple roots of the Kac-Moody algebra L_∞ , together with the positive integer multiples of the Weyl vector $\boldsymbol{\rho}$, each with multiplicity 24. Then any nonzero root $\mathbf{x} \in II_{25,1}$ of $\mathfrak{g}_{II_{25,1}}$ has multiplicity $p_{24}(1 - \frac{1}{2}\mathbf{x}^2)$. Note that the set of simple roots is characterized by the condition $\mathbf{r} \cdot \boldsymbol{\rho} = -\frac{1}{2}\mathbf{r}^2$. The Lie algebra $\mathfrak{g}_{II_{25,1}}$ was first defined by Borcherds in [9].

This result is quite astonishing because the fake Monster Lie algebra is *not* a Kac-Moody algebra due to the presence of the lightlike simple Weyl roots which violate an axiom for these algebras.(cf. [55],[50]) Nevertheless, the structure of $\mathfrak{g}_{H_{25,1}}$ resembles a Kac-Moody algebra very well. It was Borcherds' great achievement to observe that one can define generalized Kac-Moody algebras by allowing imaginary (\equiv nonpositive norm) simple roots in the defining axioms of Kac-Moody algebras [6]. To complete our first example of a Borcherds algebra we introduce a set of 24 orthonormal transverse polarization vectors ξ_i , i.e. $\xi_i \cdot \rho = 0$, $\xi_i \cdot \xi_j = \delta_{ij}$ for $1 \leq i, j \leq 24$, and consider the following generators:

$$\begin{aligned}\xi_i(-1)e^{m\rho} &\mapsto e_i(m\rho) \\ \xi_i(-1)e^{-m\rho} &\mapsto f_i(m\rho) \\ m\rho(-1) &\mapsto h_i(m\rho)\end{aligned}$$

for $1 \leq i \leq 24$ and $m > 0$. In addition to the relations for L_∞ we get

$$\begin{aligned}[e(\mathbf{r}), f_i(m\rho)] &= 0 \\ [f(\mathbf{r}), e_i(m\rho)] &= 0 \\ [e_i(m\rho), f_j(n\rho)] &= \delta_{mn}\delta_{ij}h_i(m\rho) \\ [e_i(m\rho), e_j(n\rho)] &= 0 \\ [f_i(m\rho), f_j(n\rho)] &= 0 \\ [h(\mathbf{r}), e_i(m\rho)] &= -me_i(m\rho) \\ [h(\mathbf{r}), f_i(m\rho)] &= mf_i(m\rho) \\ [h_i(m\rho), e(\mathbf{r})] &= -me(\mathbf{r}) \\ [h_i(m\rho), f(\mathbf{r})] &= mf(\mathbf{r}) \\ [h_i(m\rho), e_j(n\rho)] &= 0 \\ [h_i(m\rho), f_j(n\rho)] &= 0 \\ [h(\mathbf{r}), h_i(m\rho)] &= 0 \\ [h_i(m\rho), h_j(n\rho)] &= 0\end{aligned}$$

The Cartan matrix looks as following:

$$\left(\begin{array}{c|ccc} L(\infty) & \vdots & \vdots & \dots \\ \hline \dots & (-1^{(24)})^T & 0_{24} & 0_{24} & \dots \\ \dots & (-2^{(24)})^T & 0_{24} & 0_{24} & \dots \\ \vdots & \vdots & \vdots & \ddots & \end{array} \right)$$

where 0_{24} is the 24×24 zero matrix, the 24 dimensional row vectors $-m^{(24)}, m \geq 0$, are given by

$$-m^{(24)} \equiv (-m \ \dots \ -m)$$

and $L(\infty)$ denotes the infinite-dimensional Cartan matrix for the Leech roots, i.e. with entry $\mathbf{r} \cdot \mathbf{s}$ in the \mathbf{r} th row and \mathbf{s} th column for Leech roots \mathbf{r}, \mathbf{s} .

5 Borcherds algebras \equiv generalized Kac-Moody algebras

5.1 Definition and properties

The purpose of this section is to develop the formal aspects of Borcherds algebras as presented in [7],[6],[10]. Borcherds algebras are also mentioned in [50].

Definition 3 :

Let $A = (a_{ij})$, $i, j \in I$, be a symmetric real matrix with no zero columns, possibly infinite but countable, satisfying the following properties:

(i) either $a_{ii} = 2$ or $a_{ii} \leq 0$

(ii) $a_{ij} \leq 0$ if $i \neq j$

(iii) $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$

Then the **universal generalized Kac-Moody algebra** associated to A is defined to be the Lie algebra $\hat{\mathfrak{g}}(A)$ given by the following generators and relations.

Generators: Elements e_i , f_i , h_{ij} for $i, j \in I$

Relations:

$$(1) [e_i, f_j] = h_{ij}$$

$$(2) [h_{ij}, e_k] = \delta_{ij}a_{jk}e_k, [h_{ij}, f_k] = -\delta_{ij}a_{jk}f_k$$

$$(3) (\text{ad } e_i)^{1-a_{ij}}e_j = 0, (\text{ad } f_i)^{1-a_{ij}}f_j = 0 \quad \text{if } a_{ii} = 2 \text{ and } i \neq j$$

$$(4) [e_i, e_j] = 0, [f_i, f_j] = 0 \quad \text{if } a_{ii} \leq 0, a_{jj} \leq 0 \text{ and } a_{ij} = 0$$

Let us make some remarks and list important properties of universal generalized Kac-Moody algebras.

1. There is a unique invariant bilinear form $(_, _)$ on $\hat{\mathfrak{g}}(A)$ such that $(e_i, f_j) = \delta_{ij}$; invariance and relations (1), (2) then imply $(h_{ii}, h_{jj}) = a_{ij}$.
2. If $a_{ii} = 2$ for all $i \in I$ then $\hat{\mathfrak{g}}(A)$ is the same as the ordinary Kac-Moody algebra with symmetrized Cartan matrix A . In general, $\hat{\mathfrak{g}}(A)$ has almost all the properties that ordinary Kac-Moody algebras have, and the only major difference is that generalized Kac-Moody algebras are allowed to have imaginary simple roots.
3. The Jacobi identity applied to the elements h_{ij} , e_k , f_l yields

$$[h_{ij}, h_{kl}] = \delta_{ij}(a_{jk} - a_{jl})h_{kl}$$

so that

- (a) h_{ij} lies in the centre of $\hat{\mathfrak{g}}(A)$ if $i \neq j$,
- (b) all the h 's commute with each other,
- (c) $h_{ij} = 0$ if the i th and the j th columns of A are not equal

The elements h_{ij} for which the i th and the j th columns of A are equal form a basis for an abelian subalgebra of $\hat{\mathfrak{g}}(A)$, called its **Cartan subalgebra** $\hat{\mathfrak{h}}$. In the case of ordinary Kac-Moody algebras, the i th and the j th columns of A cannot be equal unless $i = j$, so the only nonzero elements h_{ij} are those of the form h_{ii} which are usually denoted by h_i . The reason why we need the elements h_{ij} for $i \neq j$ is that $\hat{\mathfrak{g}}(A)$, so defined, is equal to its own universal central extension.

4. We can define a \mathbb{Z} -gradation of $\hat{\mathfrak{g}}(A)$ by $\deg(e_i) = -\deg(f_i) = n_i$ where $\{n_i | i \in I\}$ is a collection of positive integers with finite repetitions. The degree zero piece of $\hat{\mathfrak{g}}(A)$ is the Cartan subalgebra $\hat{\mathfrak{h}}$.
5. $\hat{\mathfrak{g}}(A)$ has an antilinear involution θ with $\theta(e_i) = -f_i$, $\theta(f_i) = -e_i$, $\theta(h_{ij}) = -h_{ji}$, called the **Cartan involution**.
6. The contravariant form $(x, y)_0 := (\theta(x), y)$ is "almost positive definite" on $\hat{\mathfrak{g}}(A)$ which means that $(x, x)_0 > 0$ whenever x is a homogeneous element of nonzero degree in $\hat{\mathfrak{g}}(A)$.
7. The **root lattice** Λ_R is defined to be the free abelian group generated by elements \mathbf{r}_i for $i \in I$, with the bilinear form given by $\mathbf{r}_i \cdot \mathbf{r}_j := a_{ij}$. The elements \mathbf{r}_i are called the **simple roots**. The universal generalized Kac-Moody algebra $\hat{\mathfrak{g}}(A)$ is Λ_R -graded by letting $\hat{\mathfrak{h}}$ have degree zero, e_i have degree \mathbf{r}_i and f_i have degree $-\mathbf{r}_i$. The root space of an element $\mathbf{r} \in \Lambda_R$ is the vector space of elements of $\hat{\mathfrak{g}}(A)$ of that degree; if \mathbf{r} is nonzero and has a nonzero root space then \mathbf{r} is called a **root** of $\hat{\mathfrak{g}}(A)$. A root \mathbf{r} is called **positive** if it is a sum of simple roots, and **negative** if $-\mathbf{r}$ is positive. Every root is either positive or negative. A root \mathbf{r} is called **real** if $\mathbf{r}^2 > 0$ and **imaginary** otherwise. In [50] there are proved a number of facts about the root systems of universal generalized Kac-Moody algebras.
8. There is a **denominator formula** for universal generalized Kac-Moody algebras. This states that

$$e^{\boldsymbol{\rho}} \prod_{\mathbf{r} > 0} (1 - e^{\mathbf{r}})^{\text{mult}(\mathbf{r})} = \sum_{w \in \mathcal{W}} \det(w) w \left(e^{\boldsymbol{\rho}} \sum_{\mathbf{r}} \epsilon(\mathbf{r}) e^{\mathbf{r}} \right)$$

Here $\boldsymbol{\rho}$ is the Weyl vector (\equiv vector with $\boldsymbol{\rho} \cdot \mathbf{r} = -\frac{1}{2}\mathbf{r}^2$ for all simple roots), $\mathbf{r} > 0$ means that \mathbf{r} is a positive root, \mathcal{W} is the Weyl group (\equiv group of isometries of Λ_R generated by the reflections $\sigma_i(\mathbf{r}) := \mathbf{r} - (\mathbf{r} \cdot \mathbf{r}_i)\mathbf{r}_i$ corresponding to the *real* simple roots \mathbf{r}_i); $\det(w)$ is defined to be $+1$ or -1 depending on whether w is the product of an even or odd number of reflections and $\epsilon(\mathbf{r})$ is $(-1)^n$ if \mathbf{r} is the sum of n distinct pairwise orthogonal imaginary simple roots, and zero otherwise.

Note that the Weyl vector $\boldsymbol{\rho}$ may be replaced by any vector having inner product $-\frac{1}{2}\mathbf{r}^2$ with all *real* simple roots \mathbf{r} since $e^{w(\boldsymbol{\rho}) - \boldsymbol{\rho}}$ only involves inner products of $\boldsymbol{\rho}$ with the real simple roots.

For ordinary Kac-Moody algebras there are no imaginary simple roots, so the sum over \mathbf{r} equals one and we end up with the well-known denominator formula.

9. There is a natural homomorphism of abelian groups from the root lattice Λ_R to the Cartan subalgebra $\hat{\mathfrak{h}}$ taking \mathbf{r}_i to $h_{ii} \equiv h_i$ which preserves the bilinear form. This map is not usually injective. It is possible for n imaginary simple roots to have the same image h_i in $\hat{\mathfrak{h}}$ in which case we say, by abuse of language, that \mathbf{r}_i is a simple root "of multiplicity n ".

If we take the quotient of a universal generalized Kac-Moody algebra we obtain a generalized Kac-Moody algebra.

Definition 4 :

A **generalized Kac-Moody algebra** is a Lie algebra \mathfrak{g} with an almost positive definite contravariant form, which means that \mathfrak{g} has the following properties:

- (1) $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is \mathbb{Z} -graded with $\dim \mathfrak{g}_i < \infty$
- (2) \mathfrak{g} has an involution θ which acts as -1 on \mathfrak{g}_0 and maps \mathfrak{g}_i to \mathfrak{g}_{-i}
- (3) \mathfrak{g} carries an invariant bilinear form $(_, _)$ invariant under θ such that $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$
- (4) The contravariant form $(x, y)_0 := -(\theta(x), y)$ is positive definite on \mathfrak{g}_i if $i \neq 0$
- (5) $\mathfrak{g}_0 \subset [\mathfrak{g}, \mathfrak{g}]$

(If the last condition is omitted we may add an abelian algebra of outer derivations to a generalized Kac-Moody algebra. If the i th and the j th column of A are equal then $\hat{\mathfrak{g}}(A)$ has an outer derivation d defined by $[d, e_i] = e_j$, $[d, e_j] = -e_i$, $[d, f_i] = f_j$, $[d, f_j] = -f_i$, and $[d, e_k] = [d, f_k] = 0$ if $k \neq i, j$. These outer derivations do not always commute with the elements of the Cartan subalgebra $\hat{\mathfrak{h}}$.)

The main theorem about generalized Kac-Moody algebras states that we can construct any generalized Kac-Moody algebra from some universal generalized Kac-Moody algebra by factoring out some of the centre and adding a commuting algebra of outer derivations.

It is easy to see that the fake Monster Lie algebra $\mathfrak{g}_{\text{II}_{25,1}}$ is a generalized Kac-Moody algebra in the sense of Definition 4. If we fix any negative norm vector \mathbf{x} of $\text{II}_{25,1}$ not perpendicular to any Leech root then we can make $\mathfrak{g}_{\text{II}_{25,1}}$ into a \mathbb{Z} -graded Lie algebra by using the inner product with \mathbf{x} as the degree. Then all the conditions of the definition of a generalized Kac-Moody algebra are satisfied for $\mathfrak{g}_{\text{II}_{25,1}}$.

5.2 Simple examples

Let us consider the two simplest examples of Borcherds algebras, namely, the Borcherds extensions of $\widehat{\mathfrak{su}(2)}$ and $\widehat{\mathfrak{su}(2)}$ with one lightlike simple root.(see [66])

We start with the following generalized Cartan matrix:

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

The simple roots \mathbf{r}_1 and \mathbf{r}_2 are imaginary and real, respectively. The scalar product of two roots $\mathbf{r} = m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2$, $\mathbf{s} = n_1 \mathbf{r}_1 + n_2 \mathbf{r}_2$ is given by $\mathbf{r} \cdot \mathbf{s} = -m_1 n_2 - m_2 n_1 + 2m_2 n_2$. A presentation of the corresponding Borcherds algebra $\hat{\mathfrak{g}}(A)$ is given in terms of six generators $e_1, e_2, f_1, f_2, h_1, h_2$, and relations,

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, e_j] &= a_{ij} e_j \\ [h_i, f_j] &= -a_{ij} f_j \\ [e_2, [e_2, e_1]] &= 0 \\ [f_2, [f_2, f_1]] &= 0 \end{aligned}$$

We define the **fundamental dominant weights** λ_1, λ_2 relative to the simple roots $\mathbf{r}_1, \mathbf{r}_2$ by the dual base condition

$$\lambda_i \cdot \mathbf{r}_j \stackrel{!}{=} \delta_{ij} \quad i, j = 1, 2$$

so that $\lambda_1 = -2\mathbf{r}_1 - \mathbf{r}_2$ and $\lambda_2 = -\mathbf{r}_1$. An important result of representation theory (see [49],[71],[35], e.g.) then tells us that any highest weight associated with a highest weight representation can be written as a sum of fundamental dominant weights.

To actually compute the weight multiplicities for a highest weight representation of a Borcherds algebra one derives from the denominator formula some useful recursion formulas which can be put on a computer. A part of the result for the $(1, 0)$ fundamental representation (i.e. with λ_1 as highest weight) of $\widehat{\mathfrak{g}}(A)$ is listed in the following table:

n_1	multiplicity of weight $\lambda = n_1\mathbf{r}_1 + [n_2]\mathbf{r}_2$	$\mathfrak{su}(2)$ content	as tensor products
0	[0]	(0)	(0)
1	[1] + 1[0]	(1)	(1)
2	[2] + 2[1] + 1[0]	(2) + 1(0)	(1) \otimes (1)
3	[3] + 3[2] + 3[1] + 1[0]	(3) + 2(1)	(1) \otimes (1) \otimes (1)
4	[4] + 4[3] + 6[2] + 4[1] + 1[0]	(4) + 3(2) + 2(0)	(1) \otimes (1) \otimes (1) \otimes (1)
:	:	:	:

Note that we slice the representation with the imaginary simple root \mathbf{r}_1 which means that we regard n_1 as a number operator eigenvalue. At each level n_1 we can rewrite the portion of the representation in terms of $\mathfrak{su}(2)$ representations with highest weight $\Lambda(h_2) = 2l$. For example, at slice $n_1 = 3$ we find weights $\lambda = n_1\mathbf{r}_1 + n_2\mathbf{r}_2$ with $n_2 = 0, 1, 2, 3$ and multiplicity 1, 3, 3, 1, respectively. Thus we have one four-dimensional (iso)spin $\frac{3}{2}$ and two two-dimensional (iso)spin $\frac{1}{2}$ representations indicated by 2(1) + 1(3) in the table. And this is nothing but the tensor product of three (iso)spin $\frac{1}{2}$. (see also the review [67])

We observe that the multiplicity of the weight $\lambda = n_1\mathbf{r}_1 + n_2\mathbf{r}_2$ equals $\binom{n_1}{n_2}$, the total number of states at slice n_1 being $\sum_{n_2=0}^{n_1} \binom{n_1}{n_2} = 2^{n_1}$ which is the coefficient of q^{n_1} in the partition function $P(q) = \frac{1}{1-2q}$. This shows that the $\mathfrak{su}(2)$ structure at slice n_1 is the tensor product of the two-dimensional (iso)spin $\frac{1}{2}$ representation with itself n_1 times with no symmetry or antisymmetry constraints.

The number operator, N , and the diagonalized operator of $\mathfrak{su}(2)$, I_3 , can be expressed in terms of the $\{h_1, h_2\}$ basis of the Cartan subalgebra as $N = -2h_1 - h_2$ and $2I_3 = h_2$, respectively, which shows that the operator N counting the number of e_1 operators lies in the Cartan subalgebra and corresponds to the root $\mathbf{r}_N = \lambda_1 = -2\mathbf{r}_1 - \mathbf{r}_2$.

In the case of extending the affine Lie algebra $\widehat{\mathfrak{su}(2)}$ we consider the following Cartan matrix:

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

The roots are of the form $\mathbf{r} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3$. It is most natural to look break the $(1, 0, 0)$ representation of the corresponding Borcherds algebra $\widehat{\mathfrak{g}}(A)$ into representations of $\widehat{\mathfrak{su}(2)}$, i.e. again we slice the representation with the imaginary simple root \mathbf{r}_1 . Computer calculations for the first values of n_1 show that the $\widehat{\mathfrak{su}(2)}$ structure at slice n_1 is the tensor product of the $(1, 0)$

fundamental representation of $\widehat{\mathfrak{su}(2)}$ with itself n_1 times. In other words, starting from two-dimensional current algebra, the fundamental $(1, 0, 0)$ representation of the simplest Borcherds extension contains a vacuum at $n_1 = 0$, single particles at $n_1 = 1$, two particle states at $n_2 = 2$, and so on. The surprising result is that the full multiparticle space of states is included in this single representation. Moreover both number operator and Hamilton operator are members of the Cartan subalgebra: $N = -h_2 - h_3$, $L_{(0)} = -h_1 - h_2 - \frac{1}{2}h_3$, $2I_3 = h_2$.

To give an interpretation and a possible application of this feature we make a short digression to the question of symmetry in quantum theory. We will follow [3] and [73].

Consider the differential operator $W = i\frac{\partial}{\partial t} - H$ where H denotes the Hamilton operator of some nonrelativistic quantum system. We define wavefunctions ψ of the system as solutions of the Schrödinger equation $W\psi = 0$. If there are operators G_j , $j = 1, \dots, n$, forming a Lie algebra \mathfrak{g} and satisfying, on the space Φ of solutions, $[W, G_j]\psi = 0$ then Φ is a representation space for the **dynamical Lie algebra** \mathfrak{g} of the quantum system. In general, \mathfrak{g} contains time-dependent operators $G(t)$ satisfying the Heisenberg equation

$$[i\frac{\partial}{\partial t}, G(t)]\psi = [H, G(t)]\psi \quad \psi \in \Phi$$

The subalgebra $\mathfrak{g}' := \{G_j \in \mathfrak{g} \mid [G_j, H] = 0\}$ is a more narrow definition of symmetry. The **maximal symmetry algebra of H** is defined to be the subalgebra $\mathfrak{g}'' \subset \mathfrak{g}'$ of time-independent operators commuting with the Hamilton operator.

The Heisenberg equation has the solution $G(t) = e^{itH}G(0)e^{-itH}$ where the evolution operator e^{itH} clearly commutes with the energy operator H . Hence the time-dependent dynamical Lie algebra $\mathfrak{g} = \{G_j(t) \mid j = 1, \dots, n\}$ and the time-independent dynamical Lie algebra $\{G_j(0) \mid j = 1, \dots, n\}$ are unitarily equivalent which allows us to restrict ourselves in concrete problems of the analysis of time-independent dynamical Lie algebras.

For stationary solutions of the Schrödinger equation of the form $\psi(t) = e^{-iEt}u$ we obtain the eigenvalue equation $Hu = Eu$. An eigenspace of H for a fixed value E of the energy is already a representation space of the maximal symmetry algebra \mathfrak{g}'' of H . Hence \mathfrak{g}'' should be rather called "algebra of degeneracy of the energy". In order to solve the quantum mechanical problem completely, we still have to determine the spectrum of H .

As an example let us analyze the spectrum of the nonrelativistic hydrogen atom. Rotational symmetry of the Hamilton operator, i.e. $[H, L_i] = 0$, $i = 1, 2, 3$, suggests $\mathfrak{so}(3)$ as symmetry algebra leading to a $2l + 1$ -fold degeneracy for each energy level. However, this is just the kinematical symmetry algebra of the hydrogen atom. It turns out that each energy eigenvalue E_n has multiplicity n^2 (neglecting spin) independent of the angular momentum quantum number l . For a given principal quantum number n , the eigenspace \mathcal{H}_n of E_n can be decomposed into $2l + 1$ -dimensional irreducible representations $\mathcal{D}(l)$ of $\mathfrak{so}(3)$:

$$\begin{aligned} \mathcal{H}_n &= \bigoplus_{l=0}^{n-1} \mathcal{D}(l) \\ \dim \mathcal{H}_n &= \sum_{l=0}^{n-1} (2l + 1) = n^2 \end{aligned}$$

This additional degeneracy is surprising and can only be explained in terms of a higher ("hidden") symmetry of the Hamilton operator. In fact, Pauli showed that the classical Runge-Lenz vector which occurs as a constant of motion in the classical Kepler problem, leads to three

hermitian quantum-mechanical operators commuting with the Hamilton operator. We conclude that the maximal symmetry algebra of H is the Lie algebra $\mathfrak{so}(4)$, i.e. for a given n , the eigenspace \mathcal{H}_n of $E_n < 0$ (bound states) carries a single n^2 -dimensional irreducible representation of $\mathfrak{so}(4)$ (For $E_n > 0$, the continuum states, we have $\mathfrak{so}(3, 1)$). Thus $\mathfrak{so}(4)$ may be interpreted as the degeneracy algebra of the nonrelativistic hydrogen atom.

If the states of the hydrogen atom are labeled by the traditional quantum numbers $|nlm\rangle$ associated with the solution in spherical coordinates then, in constructing the full dynamical algebra, we must find an algebra which contains $\mathfrak{so}(4)$ as a subalgebra and includes operators that ladder n and l . A careful analysis exhibits the 15-dimensional Lie algebra $\mathfrak{so}(4, 2)$ as a dynamical algebra for the hydrogen atom which means that its operators permit us to pass from any hydrogenic state $|nlm\rangle$ to any other state $|n'l'm'\rangle$. Hence there is a *single* irreducible representation of $\mathfrak{so}(4, 2)$ that covers *all* the states of the hydrogen atom. Of course this representation must be infinite-dimensional.

It is worth mentioning that already $\mathfrak{so}(4, 1)$ possesses a single irreducible representation which covers the complete set of quantum numbers n, l, m , and may therefore be regarded as the quantum-number algebra of the hydrogen atom. The enlargement of $\mathfrak{so}(4, 1)$ to $\mathfrak{so}(4, 2)$ introduces no additional quantum numbers and leaves the representation space unchanged but it requires additional operators that can be identified with interaction operators. Thus, in principle, the calculation of electromagnetic transition amplitudes and the Stark effect has been reduced to an algebraic calculation without any need to compute integrals. It is also remarkable that the generators of $\mathfrak{so}(4, 2)$ may be realized in terms of the four-dimensional Dirac γ -matrices.

After this digression on symmetries it is tempting to speculate about applications of Borcherds algebras in physics. It might be possible to construct quantum field theories in which a Borcherds algebra plays the role of a sort of dynamical Lie algebra. One would expect to find all quantum states within a single representation. In particular, the dynamical algebra should comprise the Hamilton operator as well as operators that change number of particles. The underlying Lie algebra without Borcherds extension then could determine the maximal symmetry algebra of the Hamilton operator.

Another area where Borcherds algebras might emerge is string field theory. It is astonishing that the irreducible representations of the above discussed examples precisely match the Fock space of bosonic string field theory with the underlying Kac-Moody algebra as spectrum-generating algebra for the 1-string Hilbert space.

After discussing the simplest examples of Borcherds algebras we finally want to return to the probably least trivial example of a Borcherds algebra, namely, the Monster Lie algebra invented by Borcherds in [10].

5.3 The Monster Lie algebra

In [32], Frenkel, Lepowsky and Meurman constructed the Monster vertex algebra which is acted on by the Monster simple sporadic group. The underlying vector space which is called **Moonshine Module** [31] and is denoted by \mathcal{F}^\natural provides a natural infinite-dimensional representation of the Monster is characterized by the following properties:

- (i) \mathcal{F}^\natural is a vertex operator algebra with a conformal vector ω of dimension 24 and a positive definite bilinear form

(ii) $\mathcal{F}^\natural = \bigoplus_{n \geq -1} \mathcal{F}_n^\natural$ where $\mathcal{F}_n^\natural \equiv \mathcal{F}_{(n+1)}^\natural$ is the eigenspace of $L_{(0)}$ with eigenvalue $n+1$, and the dimension of \mathcal{F}_n^\natural is given via the generating function

$$\sum_{n \geq -1} \dim \mathcal{F}_n^\natural q^n = J(q) \equiv j(q) - 744 = q^{-1} + 196884q + \dots$$

(iii) The Monster group acts on \mathcal{F}^\natural preserving the vertex operator algebra structure, the conformal vector ω and the bilinear form

(Note the shift of weights!) It is crucial that the constant term of $J(q)$ is zero which entails that there are no primary fields of weight one. Remember that this property was essential for constructing the Griess algebra of primary fields of weight two in Subsection 3.5. Indeed, \mathcal{F}_1^\natural coincides with the Griess algebra since additionally the spectrum of $L_{(0)}$ is nonnegative and $\mathcal{F}_{-1}^\natural$ is one-dimensional.

The Monster vertex algebra is realized explicitly as

$$\mathcal{F}^\natural = \mathcal{F}_{\Lambda_{\text{Leech}}}^+ \oplus \mathcal{F}_{\Lambda_{\text{Leech}}}^{\prime+}$$

where $\mathcal{F}_{\Lambda_{\text{Leech}}}$ denotes the vertex operator algebra associated with the Leech lattice, the unique 24-dimensional even unimodular Euclidian lattice with no elements of square length two. The symbol '+' denotes the subspace fixed by a certain involution whereas the prime indicates a twisted construction involving the square roots of the formal variables. A readable account to the Monster module as \mathbb{Z}_2 -orbifold of a bosonic string theory compactified to the Leech lattice can be found in [69] whereas a \mathbb{Z}_p -orbifold description is presented in [26]. It is interesting that there is also an approach to the Monster module based on twisting the heterotic string. [44]

The starting point for the definition of a Monster Lie algebra should be the fake Monster Lie algebra. We use the fact that the Lorentzian lattice $\mathbb{II}_{25,1}$ can be written as the direct sum of the Leech lattice and the unique two-dimensional even unimodular Lorentzian lattice $\mathbb{II}_{1,1}$. The elements of $\mathbb{II}_{1,1}$ are usually represented as pairs $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$ with inner product matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ so that $(m, n)^2 = -2mn$.

We need the following general result on vertex algebras [28]. Given two vertex algebras $(\mathcal{F}_i, \mathcal{V}_i, \mathbf{1}_i, \omega_i)$, $i = 1, 2$, the vector space

$$\mathcal{F} := \mathcal{F}_1 \otimes \mathcal{F}_2$$

becomes a vertex algebra of rank equal to the sum of the ranks of the \mathcal{F}_i when we provide \mathcal{F} with the tensor product grading and set

$$\begin{aligned} \mathcal{V}(\psi_1 \otimes \psi_2, z) &:= \mathcal{V}_1(\psi_1, z) \otimes \mathcal{V}_2(\psi_2, z) \quad \text{for } \psi_i \in \mathcal{F}_i \\ \mathbf{1} &:= \mathbf{1}_1 \otimes \mathbf{1}_2 \\ \omega &:= \omega_1 \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes \omega_2 \end{aligned}$$

In our case we obtain

$$\mathcal{F}_{\mathbb{II}_{25,1}} = \mathcal{F}_{\Lambda_{\text{Leech}} \oplus \mathbb{II}_{1,1}} = \mathcal{F}_{\Lambda_{\text{Leech}}} \otimes \mathcal{F}_{\mathbb{II}_{1,1}}$$

and the vertex algebra associated with the Lorentzian lattice $\mathbb{II}_{25,1}$ is the tensor product of the vertex algebras corresponding to $\mathcal{F}_{\Lambda_{\text{Leech}}}$ and $\mathcal{F}_{\mathbb{II}_{1,1}}$. One finds that the Leech lattice gives rise to a vertex operator algebra with conformal vector of dimension 24 and a positive definite bilinear

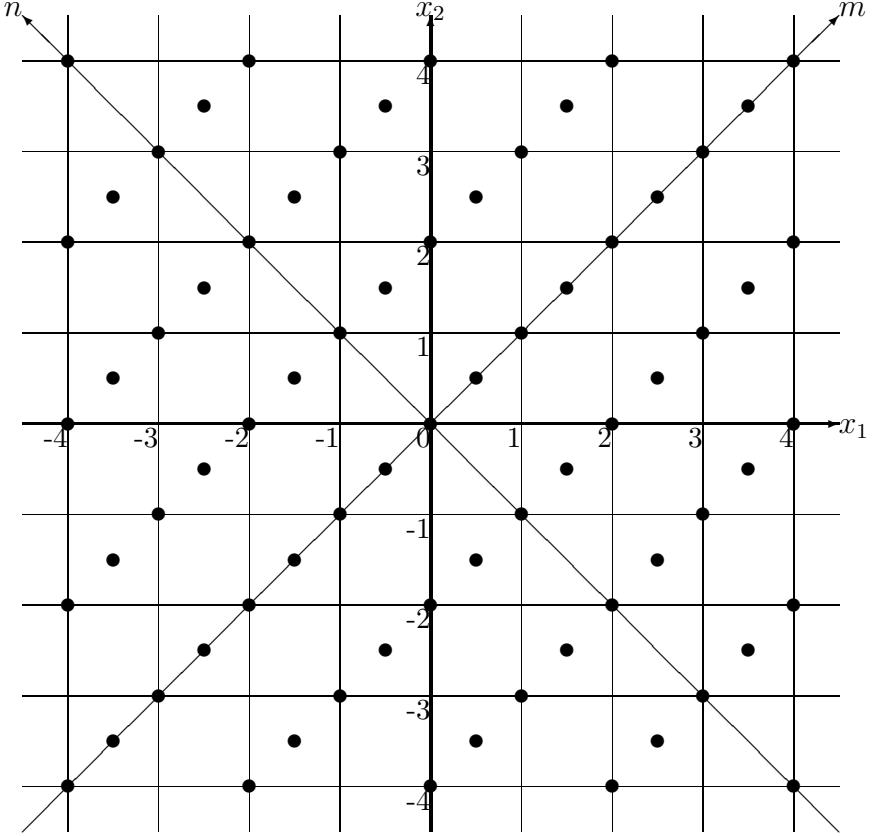


Figure 2: The Lorentzian lattice $\mathbb{II}_{1,1}$

form. Furthermore, $\mathcal{F}_{\Lambda_{\text{Leech}}} = \bigoplus_{n \geq -1} \mathcal{F}_n^{\Lambda_{\text{Leech}}}$ where $\mathcal{F}_n^{\Lambda_{\text{Leech}}} \equiv \mathcal{F}_{(n+1)}^{\Lambda_{\text{Leech}}}$ is the eigenspace of $L_{(0)}$ with eigenvalue $n+1$, and the dimension of $\mathcal{F}_n^{\Lambda_{\text{Leech}}}$ is given via the generating function

$$\sum_{n \geq -1} \dim \mathcal{F}_n^{\Lambda_{\text{Leech}}} q^n = J(q) + 24 \equiv j(q) - 720 = q^{-1} + 24 + 196884q + \dots$$

The striking similarity between $\mathcal{F}_{\Lambda_{\text{Leech}}}$ and the Moonshine module \mathcal{F}^\natural suggests the construction of the tensor vertex algebra corresponding to $\mathcal{F}^\natural \otimes \mathcal{F}_{\mathbb{II}_{1,1}}$. Then the Monster Lie algebra $\mathfrak{g}^{\natural \otimes \mathbb{II}_{1,1}}$ should be defined as the subspace

$$\mathfrak{g}^{\natural \otimes \mathbb{II}_{1,1}} := \mathcal{P}_{(1)}^{\natural \otimes \mathbb{II}_{1,1}} / \text{kernel}(_, _)_{\natural \otimes \mathbb{II}_{1,1}}$$

Obviously, $\mathfrak{g}^{\natural \otimes \mathbb{II}_{1,1}}$ is $\mathbb{II}_{1,1}$ -graded, carries an invariant bilinear form and has an involution which is induced by the trivial automorphism of \mathcal{F}^\natural and the natural involution θ of $\mathcal{F}_{\mathbb{II}_{1,1}}$ (acting as -1 on $\mathbb{II}_{1,1}$ and on the piece of degree $\mathbf{0} \in \mathbb{II}_{1,1}$ of $\mathcal{F}_{\mathbb{II}_{1,1}}$).

We can apply the no-ghost theorem of Goddard and Thorne in a modified version of Borcherds to say more about the Monster Lie algebra.

Theorem 6 :

Suppose that

- (i) $\mathcal{F} = \bigoplus_{n \geq -1} \mathcal{F}_n$ is a vertex operator algebra of rank 24 where $\mathcal{F}_n \equiv \mathcal{F}_{(n+1)}$ is the piece of conformal weight $n+1$ (so that $L_{(0)}$ has nonnegative spectrum)

(ii) \mathcal{F} is equipped with a nonsingular bilinear form $(_, _)_{\mathcal{F}}$ such that the adjoint of $L_{(n)}$ is $L_{(-n)}$

(iii) \mathcal{F} is acted on by a group G which preserves all this structure

Then we have the following natural G module isomorphisms:

$$\begin{aligned} \mathcal{P}_{(1),(\mathbf{r})}^{\otimes \Pi_{1,1}} / \text{kernel}(_, _)_{\mathcal{F} \otimes \mathcal{F}_{\Pi_{1,1}}} &\cong \mathcal{F}_{-\frac{1}{2}\mathbf{r}^2} \equiv \mathcal{F}_{(1-\frac{1}{2}\mathbf{r}^2)} \quad \text{for } \mathbf{0} \neq \mathbf{r} \in \Pi_{1,1} \\ \mathcal{P}_{(1),(\mathbf{0})}^{\otimes \Pi_{1,1}} / \text{kernel}(_, _)_{\mathcal{F} \otimes \mathcal{F}_{\Pi_{1,1}}} &\cong \mathcal{F}_0 \oplus \mathbb{R}^2 \equiv \mathcal{F}_{(1)} \oplus \mathbb{R}^2 \end{aligned}$$

where $\mathcal{P}_{(1),(\mathbf{r})}^{\otimes \Pi_{1,1}}$ denotes the subspace of degree $\mathbf{r} \in \Pi_{1,1}$ of the physical space $\mathcal{P}_{(1)}^{\otimes \Pi_{1,1}} = \{\psi \in \mathcal{F} \otimes \mathcal{F}_{\Pi_{1,1}} | L_{(n)}\psi = \delta_{n0}\psi, n \geq 0\}$, and G acts trivially on $\mathcal{F}_{\Pi_{1,1}}$ and \mathbb{R}^2 .

Proof: See [10]

The no-ghost theorem implies that the piece of nonzero degree $(m, n) \in \Pi_{1,1}$ of the Monster Lie algebra $\mathfrak{g}^{\otimes \Pi_{1,1}}$ is isomorphic to the piece $\mathcal{F}_{mn}^{\natural}$ of the Moonshine module and that the contravariant form $(_, _)_0$ is positive definite on that piece of $\mathfrak{g}^{\otimes \Pi_{1,1}}$. The degree zero piece of $\mathfrak{g}^{\otimes \Pi_{1,1}}$ is isomorphic to \mathbb{R}^2 . Thus, schematically the Monster Lie algebra looks like (cf. [10])

$$\begin{array}{cccccccccc} & & & & n & & & & & \\ & & & & \uparrow & & & & & \\ \vdots & \vdots \\ \dots & 0 & 0 & 0 & 0 & \mathcal{F}_4^{\natural} & \mathcal{F}_8^{\natural} & \mathcal{F}_{12}^{\natural} & \mathcal{F}_{16}^{\natural} & \dots \\ \dots & 0 & 0 & 0 & 0 & \mathcal{F}_3^{\natural} & \mathcal{F}_6^{\natural} & \mathcal{F}_9^{\natural} & \mathcal{F}_{12}^{\natural} & \dots \\ \dots & 0 & 0 & 0 & 0 & \mathcal{F}_2^{\natural} & \mathcal{F}_4^{\natural} & \mathcal{F}_6^{\natural} & \mathcal{F}_8^{\natural} & \dots \\ \dots & 0 & 0 & 0 & \mathcal{F}_{-1}^{\natural} & 0 & \mathcal{F}_1^{\natural} & \mathcal{F}_2^{\natural} & \mathcal{F}_3^{\natural} & \mathcal{F}_4^{\natural} & \dots \\ \hline 0 & 0 & 0 & 0 & \mathbb{R}^2 & 0 & 0 & 0 & 0 & \longrightarrow m \\ \dots & \mathcal{F}_4^{\natural} & \mathcal{F}_3^{\natural} & \mathcal{F}_2^{\natural} & \mathcal{F}_1^{\natural} & 0 & \mathcal{F}_{-1}^{\natural} & 0 & 0 & \dots \\ \dots & \mathcal{F}_8^{\natural} & \mathcal{F}_6^{\natural} & \mathcal{F}_4^{\natural} & \mathcal{F}_2^{\natural} & 0 & 0 & 0 & 0 & \dots \\ \dots & \mathcal{F}_{12}^{\natural} & \mathcal{F}_9^{\natural} & \mathcal{F}_6^{\natural} & \mathcal{F}_3^{\natural} & 0 & 0 & 0 & 0 & \dots \\ \dots & \mathcal{F}_{16}^{\natural} & \mathcal{F}_{12}^{\natural} & \mathcal{F}_8^{\natural} & \mathcal{F}_4^{\natural} & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots \end{array}$$

If we finally define an element of $\mathfrak{g}^{\otimes \Pi_{1,1}}$ of degree $(m, n) \in \Pi_{1,1}$ to have \mathbb{Z} -degree $2m + n$ then, with this \mathbb{Z} -grading, the Monster Lie algebra is seen to be a generalized Kac-Moody algebra.

The $\Pi_{1,1}$ -grading of $\mathfrak{g}^{\otimes \Pi_{1,1}}$ looks like a root space decomposition for the Monster Lie algebra and this suspicion turns out to be true. To see this we go the other way round and try to find out what the Borcherds algebra with root lattice $\Pi_{1,1}$ might be. Let us start with the two vectors $\pm(1, -1) \in \Pi_{1,1}$ of square length two one of which should be chosen as a real simple root, say, $(1, -1)$. Then we may take $\rho := (-1, 0)$ as a (lightlike) Weyl vector so that the remaining (imaginary!) simple roots $\mathbf{r} = (m, n)$ are determined by the condition $\rho \cdot \mathbf{r} \stackrel{!}{=} -\frac{1}{2}\mathbf{r}^2$ ($\iff n = mn$). Moreover we know that simple roots must have non-positive inner product with each other. We conclude that the vectors $(1, n)$, $n = -1$ or $n > 0$, constitute a set of simple roots for the root lattice $\Pi_{1,1}$.

The denominator formula for the corresponding Borcherds algebra then reads

$$p^{-1} \prod_{\substack{m > 0 \\ n \in \mathbb{Z}}} (1 - p^m q^n)^{\text{mult}(m, n)} = \sum_{w \in \mathcal{W}} \det(w) w \left(p^{-1} \left(1 - \sum_{n > 0} \text{mult}(1, n) p q^n \right) \right)$$

where we write p and q for the elements $e^{(1,0)}$ and $e^{(0,1)}$ of the group ring of $\mathbb{II}_{1,1}$, respectively. Also note that all the imaginary simple roots $(1, n)$, $n \geq 1$, have nonzero inner product with each other so that there is no extra contribution of pairwise orthogonal imaginary simple roots on the right-hand side. Since the lattice $\mathbb{II}_{1,1}$ has only one real simple root the Weyl group has order two and is generated by the corresponding reflection which exchanges p and q ($\sigma_{(1,-1)}p = e^{(1,0)-\{(1,-1)\cdot(1,0)\}(1,-1)} = q$). Hence

$$\begin{aligned} p^{-1} \prod_{\substack{m>0 \\ n \in \mathbb{Z}}} (1 - p^m q^n)^{\text{mult}(m,n)} &= p^{-1} - \sum_{n>0} \text{mult}(1, n) q^n - \left(q^{-1} - \sum_{n>0} \text{mult}(1, n) p^n \right) \\ &= \sum_{\substack{n \geq -1 \\ n \neq 0}} \text{mult}(1, n) (p^n - q^n) \end{aligned}$$

where we used the fact that real simple roots always have multiplicity one. Borcherds [10] was able to determine the unknown root multiplicities by establishing an identity for the elliptic modular function $j(q)$ which turned out to be precisely the above denominator formula:

$$p^{-1} \prod_{\substack{m>0 \\ n \in \mathbb{Z}}} (1 - p^m q^n)^{c_{mn}} = j(p) - j(q)$$

with $j(q) - 744 = \sum_{n \geq -1} c_n q^n = q^{-1} + 196884q + \dots$

We summarize: The simple roots of the Monster Lie algebra $\mathfrak{g}^{\natural \otimes \mathbb{II}_{1,1}}$ are the vectors $(1, n)$, $n = -1$ or $n \geq 1$, each with multiplicity c_n .

6 Omissions and Outlook

In an introductory exposition of a rapidly developing area like vertex algebras there are necessarily left out many interesting topics. In the following we want to comment briefly on the omitted subjects of the theory of vertex algebras as well as on the latest achievements.

First of all the formalism can be extended in order to include **twisted vertex operators** which involve square roots of the formal variables. They are essential for constructing twisted representations of affine Lie algebras. Moreover, as already mentioned, the construction of the Moonshine module relies heavily on twisting vertex operators. For this purpose the whole framework of formal calculus has to be slightly modified such that it is also true for non-integral powers of the formal variables [32]. In physics literature a similar treatment can be found in [18] and [19].

Another important issue is **representation theory of vertex algebras**. Categorical notions such as module, homomorphism, irreducibility, simplicity etc. have to be defined properly in the new context. One is led naturally to the definition of rational vertex operator algebras which correspond to the familiar rational conformal field theories in physics. The analogues of chiral vertex operators in conformal field theory [57] are precisely the intertwining operators for vertex operator algebras introduced in [28]. Therefore the notion of fusion rules for vertex operator algebras also arises. At present considerable effort is spent on representation theory of vertex algebras associated with even lattices and in particular on the study of the Moonshine module [22], [21], [20], [25].

Recently, Schellekens [60] succeeded in classifying $c = 24$ self-dual chiral bosonic meromorphic conformal field theories (see also [54], [61]). His analysis was based upon modular invariance of the partition function on the torus. Hence we can ask how this result fits into the

concept of vertex algebras. However, as presented so far, there is no such thing like a modular parameter in the framework of vertex algebras which just mirrors the fact that we have dealt essentially with meromorphic conformal field theory on the Riemann sphere. Three years ago Zhu [74] introduced **correlation functions on the torus** for vertex operator algebras and found that, by taking graded dimensions of vertex operators of certain vertex operator algebras, one obtains indeed modular forms. In particular, Zhu established modular invariance of the characters of the irreducible representations as a general property of vertex operator algebras. Translated into physics language this means that properties of conformal field theory on the Riemann sphere directly imply the corresponding properties on the torus.

Motivated by the question how to place the theory of Z-algebras and parafermion algebras into an elegant axiomatic context and to embed them into more natural algebras Dong and Lepowsky developed the theory of vertex algebras much further. They defined generalized vertex operator algebras and generalized vertex algebras and finally introduced the even more general notion of **abelian intertwining algebras** where one-dimensional braid group representations are incorporated intrinsically into the algebraic structure of vertex algebras and their modules [23]. This new avenue in the formulation of vertex algebras is presented carefully in the monograph [24].

Besides the “standard” approach to two-dimensional conformal quantum field theory via states, fields, and operator products [4] there is also a nice geometric formulation in terms of punctured Riemann surfaces and sewing operations ([70],[1] or [72] and references therein). In the language of modular functors this was made mathematically rigorous by Segal [62], [63]. In the spirit of this geometric concept Huang [46], [45], [47] introduced **geometric vertex operator algebras** whose essential ingredients are the family of moduli spaces of n -punctured Riemann spheres and the operations of sewing those spheres together. He was able to prove the (categorical) equivalence of that definition with the familiar notion of vertex operator algebras (Definition 2) thereby delivering an intrinsic geometric interpretation of vertex operator algebras.

This relation was exploited even further in [48] where it was argued that the moduli spaces of punctured Riemann spheres equipped with the sewing operation constitute an algebraic structure which is called analytic associative \mathbb{C}^\times -rescalable partial operad. Thus vertex operator algebras may be reformulated in terms of **partial operads**. In this new language Stasheff [68] recently also established connections to homotopy Lie algebras, Gerstenhaber algebras [59], and Zwiebach’s closed string field theory [76],[75].

Motivated by recent progress in closed string field theory and in 2D string theory Moore [56] undertook a renewed investigation of the large symmetry algebras appearing in string theory. He considered toroidal compactifications of all spacetime coordinates (cf. Section 4.2!) and constructed new infinite-dimensional unbroken symmetry algebras of the target space. In this formulation Borcherds algebras emerge as examples of **enhanced symmetry points** in the moduli space of even unimodular lattices. In particular, the product of two copies of the fake Monster Lie algebra can be regarded as a maximal symmetry algebra of the theory since it corresponds to a unique point in the Narain moduli space (of conformal field theories) at which the closed bosonic string completely factorizes between left and right movers.

In his proof of Conway’s and Norton’s Moonshine conjectures for the Moonshine module Borcherds [10] exhibited a whole family of **monstrous Lie superalgebras** similar to the Monster Lie algebra. In fact, as the Monster Lie algebra corresponds to the identity element of the Monster group, Borcherds defined generalized Kac-Moody superalgebras associated with other elements of the Monster and worked out their root multiplicities by using the twisted

version of the denominator formula (cf. Section 5.1).

These remarks and comments should be sufficient to show that vertex algebras constitute a powerful framework in analyzing questions in conformal quantum field theory and maybe closed string field theory. It is a rapidly evolving area of mathematics which might provide us with some deep insight into the foundations of physics. Moreover, Borcherds algebras when interpreted as generalized symmetry algebras, seem to be the natural next step towards the formulation of a universal symmetry in string theory.

I would like to thank all participants of the seminar on generalized Kac-Moody algebras given at the II. Institute for Theoretical Physics during spring '93 where I was allowed to present part of this material. Their questions and suggestions were quite useful for compiling the final version. I am especially grateful to Professor Slodowy for many helpful comments and to Professor Nicolai for his constructive criticism and for encouraging me to write this review.

References

- [1] L. Alvarez-Gaumé, C. Gomez, G. Moore, and C. Vafa. Strings in the operator formalism. *Nuclear Physics*, **B303** :455–521, 1988.
- [2] T. Banks. Lectures on conformal field theory. In R. Slansky and G. West, editors, *The Santa Fe TASI-87*, pages 572–627, World Scientific, Singapore, 1988.
- [3] A. O. Barut and R. Rączka. *Theory of Group Representations and Applications*. World Scientific, Singapore, 1986.
- [4] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics*, **B241**:333–380, 1984.
- [5] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, and R. Varnhagen. W-algebras with two and three generators. *Nuclear Physics*, **B361**:255–289, 1991.
- [6] R. E. Borcherds. Central extensions of generalized Kac-Moody algebras. *Journal of Algebra*, **140** :330–335, 1991.
- [7] R. E. Borcherds. Generalized Kac-Moody algebras. *Journal of Algebra*, **115** :501–512, 1988.
- [8] R. E. Borcherds. The Leech lattice. *Proceedings of the Royal Society London, A* **389** :365–376, 1985.
- [9] R. E. Borcherds. The monster Lie algebra. *Advances in Mathematics*, **83** :30–47, 1990.
- [10] R. E. Borcherds. Monstrous Lie superalgebras. *Inventiones Mathematicae*, **109** :405–444, 1992.
- [11] R. E. Borcherds. Vertex algebras, Kac-Moody algebras, and the monster. *Proceedings of the National Academic Society USA*, **83** :3068–3071, 1986.
- [12] R. E. Borcherds, J. H. Conway, L. Queen, and N. J. A. Sloane. A monster Lie algebra? *Advances in Mathematics*, **53** :75–79, 1984.

- [13] P. Bowcock. Quasi-primary fields and associativity of chiral algebras. *Nuclear Physics*, **B356**:367–386, 1991.
- [14] J. H. Conway. The automorphism group of the 26-dimensional even unimodular Lorentzian lattice. *Journal of Algebra*, **80** :159–163, 1983.
- [15] J. H. Conway, R. A. Parker, and N. J. A. Sloane. The covering radius of the Leech lattice. *Proceedings of the Royal Society London, A* **380** :261–290, 1982.
- [16] J. H. Conway and N. J. A. Sloane. Lorentzian forms for the Leech lattice. *Bulletin of the American Mathematical Society*, **6** (2):215–217, 1982.
- [17] J. H. Conway and N. J. A. Sloane. Twenty-three constructions for the Leech lattice. *Proceedings of the Royal Society London, A* **381** :275–283, 1982.
- [18] L. Dolan, P. Goddard, and P. Montague. Conformal field theory of twisted vertex operators. *Nuclear Physics*, **B338** :529–601, 1990.
- [19] L. Dolan, P. Goddard, and P. Montague. Conformal field theory, triality and the monster group. *Physics Letters*, **B236**:165–178, 1990.
- [20] C. Dong. Representations of the moonshine module vertex operator algebra. 1992. to appear.
- [21] C. Dong. Twisted modules for vertex algebras associated with even lattices. 1992. to appear.
- [22] C. Dong. Vertex algebras associated with even lattices. 1992. to appear.
- [23] C. Dong and J. Lepowsky. Abelian intertwining algebras – a generalization of vertex operator algebras. 1992. to appear.
- [24] C. Dong and J. Lepowsky. Generalized vertex algebras and relative vertex operators. 1993. monograph, to appear.
- [25] C. Dong and G. Mason. Discrete series of the Virasoro algebra and the moonshine module. 1992. to appear.
- [26] C. Dong and G. Mason. On the construction of the moonshine module as a \mathbb{Z}_p -orbifold. 1992. to appear.
- [27] I. B. Frenkel. Representations of Kac-Moody algebras and dual resonance models. In *Applications of Group Theory in Theoretical Physics*, pages 325–353, American Mathematical Society, Providence, 1985. Lect. Appl. Math., Vol. 21.
- [28] I. B. Frenkel, Y. Huang, and J. Lepowsky. On axiomatic approaches to vertex operator algebras and modules. *Memoirs of the American Mathematical Society*, **104** (594):, July 1993. to appear.
- [29] I. B. Frenkel and V. G. Kac. Basic representations of affine Lie algebras and dual models. *Inventiones Mathematicae*, **62** :23–66, 1980.

- [30] I. B. Frenkel, J. Lepowsky, and A. Meurman. An introduction to the monster. In M. Green and D. Gross, editors, *Workshop on Unified String Theories*, pages 533–546, World Scientific, Singapore, 1986.
- [31] I. B. Frenkel, J. Lepowsky, and A. Meurman. A moonshine module for the monster. In J. Lepowsky, S. Mandelstam, and I. M. Singer, editors, *Vertex Operators in Mathematics and Physics - Proceedings of a Conference November 10-17, 1983*, pages 231–273, Springer, New York, 1985. Publications of the Mathematical Sciences Research Institute #3.
- [32] I. B. Frenkel, J. Lepowsky, and A. Meurman. *Vertex Operator Algebras and the Monster. Pure and Applied Mathematics Volume 134*, Academic Press, San Diego, 1988.
- [33] I. B. Frenkel, J. Lepowsky, and A. Meurman. Vertex operator calculus. In S. T. Yau, editor, *Mathematical aspects of string theory*, pages 150–188, World Scientific, Singapore, 1987. Advanced Series in Mathematical Physics – Vol.1.
- [34] I. B. Frenkel and Y. Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Mathematical Journal*, **66** (1):123–168, 1992.
- [35] W. Fulton and J. Harris. *Representation Theory, a first course*. Springer, New York, 1991.
- [36] P. Ginsparg. Course 1 – Applied conformal field theory. In E. Brézin and J. Zinn-Justin, editors, *Les Houches Session XLIX, 1988, Fields, Strings and Critical Phenomena*, pages 1–168, Elsevier, 1989.
- [37] P. Goddard. Meromorphic conformal field theory. In V. G. Kac, editor, *Infinite Dimensional Lie Algebras and Groups – Proceedings of the Conference held at CIRM, Luminy, July 4-8, 1988*, pages 556–587, World Scientific, Singapore, 1989. Advanced Series in Mathematical Physics Vol.7.
- [38] P. Goddard. Vertex operators and algebras. In G. Furlan, R. Jengo, J. C. Pati, D. W. Sciama, and Q. Shafi, editors, *Superstrings, Supergravity and Unified Theories*, pages 255–291, World Scientific, Singapore, 1986. The ICTP Series in Theoretical Physics – Vol.2.
- [39] P. Goddard and D. Olive. Algebras, lattices and strings. In J. Lepowsky, S. Mandelstam, and I. M. Singer, editors, *Vertex Operators in Mathematics and Physics – Proceedings of a Conference November 10-17, 1983*, pages 51–96, Springer, New York, 1985. Publications of the Mathematical Sciences Research Institute #3.
- [40] P. Goddard and D. Olive. Kac-Moody and Virasoro algebras in relation to quantum physics. *International Journal of Modern Physics*, **A 1** :303–414, 1986.
- [41] P. Goddard and C. B. Thorne. Compatibility of the dual Pomeron with unitarity and the absence of ghosts in the dual resonance model. *Physics Letters*, **B40** :235–238, 1972.
- [42] M. B. Green, J. H. Schwarz, and E. Witten. *Superstring Theory Vol. 1&2*. Cambridge University Press, 1988.
- [43] R. L. Griess. The friendly giant. *Inventiones Mathematicae*, **69** :1–102, 1982.

- [44] J. A. Harvey. Twisting the heterotic string. In M. Green and D. Gross, editors, *Workshop on Unified String Theories, Santa Barbara*, pages 704–720, World Scientific, Singapore, 1985.
- [45] Y. Huang. Geometric interpretation of vertex operator algebras. *Proceedings of the National Academic Society USA*, **88** :9964–9968, 1991.
- [46] Y. Huang. *On the geometric interpretation of vertex operator algebras*. PhD thesis, Rutgers University, New Brunswick, 1990.
- [47] Y. Huang. Vertex operator algebras and conformal field theory. *International Journal of Modern Physics, A* **7** (10):2109–2151, 1992.
- [48] Y. Huang and J. Lepowsky. *Vertex operator algebras and operads*. preprint hep-th/9301006, Institute for Advanced Study, Princeton, 1993.
- [49] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer, New York, 1972.
- [50] V. G. Kac. *Infinite dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.
- [51] S. Kass, R. V. Moody, J. Patera, and R. Slansky. *Affine Lie Algebras, Weight Multiplicities, and Branching Rules, Volume 1&2*. University of California Press, Berkeley, 1990.
- [52] W. Lerche, A. N. Schellekens, and N. P. Warner. Lattices and strings. *Physics Reports*, **177** :1–140, 1989.
- [53] B. H. Lian and G. J. Zuckerman. *New Perspectives on the BRST-algebraic Structure of String Theory*. preprint hep-th/9211072, Yale University, New Haven, 1992.
- [54] P. S. Montague. *Discussion of Self-Dual $c = 24$ Conformal Field Theories*. preprint DAMTP 92-35, University of Cambridge, 1992.
- [55] R. V. Moody. Root systems of hyperbolic type. *Advances in Mathematics*, **33** :144–160, 1979.
- [56] G. Moore. *Finite in All Directions*. preprint hep-th/9305139, Yale University, New Haven, 1993.
- [57] G. Moore and N. Seiberg. Classical and quantum conformal field theory. *Communications in Mathematical Physics*, **123** :177–254, 1989.
- [58] H. Niemeier. Definite quadratische Formen der Dimension 24 und Diskriminante 1. *Journal of Number Theory*, **5** :142–178, 1973.
- [59] M. Penkava and A. Schwarz. *On Some Algebraic Structures Arising in String Theory*. preprint hep-th/9212072, University of California, Davis, 1992.
- [60] A. N. Schellekens. *Meromorphic $c = 24$ Conformal Field Theories*. preprint TH-6478/92, CERN, Geneva, 1992.

- [61] A. N. Schellekens. *Seventy Relatives of the Monster Module*. preprint hep-th/9304098, NIKHEF–H, Amsterdam, 1993.
- [62] G. Segal. The definition of conformal field theory. In K. Bleuler and M. Werner, editors, *Differential geometrical methods in theoretical physics. Proceedings, NATO advanced research workshop, 16th international conference, Como*, pages 165–172, Kluwer, 1988.
- [63] G. Segal. Two-dimensional conformal field theories and modular functors. In B. Simon, A. Truman, and I. M. Davies, editors, *IXth International Congress on Mathematical Physics, 17–27 July 1988, Swansea, Wales*, pages 22–37, Adam Hilger, Bristol, 1989.
- [64] J. Serre. *A Course in Arithmetics*. Springer, New York, 1973.
- [65] R. Slansky. Affine Kac-Moody algebras and their representations. *Comments on Nuclear and Particle Physics*, **18** :175–214, 1988.
- [66] R. Slansky. *An Algebraic Role for Energy and Number Operators for Multiparticle States*. preprint LA-UR-91-3562, Los Alamos National Laboratory, 1991.
- [67] R. Slansky. Group theory for unified model building. *Physics Reports*, **79** :1–128, 1981.
- [68] J. Stasheff. *Closed string field theory, strong homotopy Lie algebras and the operad actions of moduli spaces*. preprint hep-th/9304061, University of North Carolina, 1993.
- [69] M. P. Tuite. Monstrous moonshine from orbifolds. *Communications in Mathematical Physics*, **146** :277–309, 1992.
- [70] C. Vafa. Conformal theories and punctured surfaces. *Physics Letters*, **B199** (2):195–202, 1987.
- [71] V. S. Varadarajan. *Lie Algebras and their Representations*. Prentice-Hall, Englewood Cliffs, 1974.
- [72] P. C. West. String vertices and induced representations. *Nuclear Physics*, **B320**:103–134, 1989.
- [73] B. G. Wybourne. *Classical Groups for Physicists*. John Wiley & Sons, New York, 1974.
- [74] Y. Zhu. *Vertex Operator Algebras, Elliptic Functions, and Modular Forms*. PhD thesis, Yale University, New Haven, 1990.
- [75] B. Zwiebach. *Closed String Field Theory: An Introduction*. preprint hep-th/9205026, MIT, Cambridge, 1993.
- [76] B. Zwiebach. *Closed String Field Theory: Quantum Action and the B-V Master Equation*. preprint hep-th/9206084, Institute for Advanced Study, Princeton, 1992.